Operator theory on holomorphic function spaces

Operator theory on holomorphic function spaces

Shift and integral operators

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LUND UNIVERSITY

DOCTORAL THESIS

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Abstract

In this thesis we study the backward shift operator on various spaces of analytic functions, and integral operators of Cesàro type also acting on various spaces of analytic functions.

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First some inspirational figures. Victor was my lecturer in Discrete mathematics back when I was still an engineering student, I loved that course. There, we finally *played* with mathematics, as one should. Marcus was my lecturer in Complex analysis. *"Weird people, these analysts"*, I thought. That course was probably what first made me seriously consider pursuing a PhD in pure mathematics. Johan, my engineering Master thesis advisor, must also be mentioned. I think we did some cool work in algebra together, and sometimes I wish there was time and opportunity for more.

Alexandru has many great qualities as an advisor, and it was a joy for me to develop as an operator theorist while working together. There were two aspects of our collaboration which were particularly important to me. One was that it was always understood that *reading a paper* meant to scroll down to the main result and try to understand which trick is used in the proof by merely glancing at it. This saved a lot of time and had no drawbacks. On a more serious note, already from the start, I felt that my own contributed thoughts and ideas were always given (due or undue) consideration, in spite of the vast difference in our experience. For this, and other things, I thank him sincerely.

Of course, a special mention must go to all the other doctoral students, current and former. Many of us met years before we all gathered up in the doctoral programme in Lund, and when we did, we made our side of the fifth floor very lively. Old gang member Jonathan had started working recently, and moved in to an office on the fourth floor. So the fourth floor also was made rather lively. I would here like to take the opportunity to forgive Anna-Maria and Yacin, the two who seemed most affected by our loud and nonsensical discussions in the corridors. I thank Adem, for our work together and for lengthy and inspirational discussions on various topics of often questionable importance. I thank Tien basically for being Tien and her unparalleled ability to reach new lows with her cringy jokes. I thank Daniele for all the laughs and activities outside of work. I thank our common friend in exile Johan (of Växjö) for... I don't even know what he could be thanked for, but he must be mentioned. To all of you, and the rest of the doctoral gang, I wish good luck with the rest of your education and I hope you will be proud of yourselves at the conclusion of your doctoral work. Oh, and I must not forget Jens, who is practically an honorary member of the doctoral student gang. Your perspective is very valuable, I think the other PhD students will agree. Lastly, and I hope he won't mind being mentioned in the same paragraph as the doctoral students, I want to thank Erik for the hard work he puts into bettering the institution.

Na koniec, pozdrowienia dla najbliższej rodziny. Mama, Tato, i Kuba. Praca doktorska dedykowana jest oczywiście wam.

Bartek

Contents

Preface

Preface

1 Summary

The main part of my PhD thesis consists of two independent research projects, both on the topic of operator theory. The results of the first of the projects, on certain integral operators called generalized Cesàro operators, are contained in the following two articles.

[**Paper I**] B. Malman, *Spectra of generalized Cesàro operators acting on growth spaces*, Integral Equations and Operator Theory, 90 (2018), p. 26.

[**Paper II**] A. Limani and B. Malman, *Generalized Cesàro operators: geometry of spectra and quasi-nilpotency*, accepted for publication in International Mathematics Research Notices (2020).

The efforts of the second project, on the backward shift operator, spawned the next two articles.

[**Paper III**] A. Aleman and B. Malman, *Density of disk algebra functions in de Branges–Rovnyak spaces*, Comptes Rendus Mathematique, 355 (2017), pp. 871–875.

[**Paper IV**] A. Aleman and B. Malman, *Hilbert spaces of analytic functions with a contractive backward shift*, Journal of Functional Analysis 277.1 (2019): 157-199.

Additionally, there is a short note on a topic different from the main theme of the two above projects, with a result which I find interesting enough to include here.

[**Paper V**] B. Malman, *Nearly invariant subspaces of de Branges spaces*, arXiv:1805.11842, (2019)

Here in this introduction I will present some background material and state most important results of my thesis. For convenience, in some cases the results will be presented in a simplified form compared to what appears in the articles. I will also discuss some problems that remain unsolved and possible directions for further work.

2 Generalized Cesàro operators

2.1 Background My first research project pertains to a class of integral operators often called generalized Cesàro operators, although in the literature the alternative name Volterra-type operators also appears. The results are contained in [**Paper I**] and [**Paper II**], the second written in collaboration with my colleague Adem Limani. The operators are of the form

$$T_g f(z) = \int_0^z g'(\zeta) f(\zeta)\, d\zeta,$$

where g and f are suitable analytic functions defined in the unit disk $\mathbb{D}$. The function g is said to be the symbol of the operator T_g, and the operator is to act on some space of analytic functions which f is a member of. Some authors work with an essentially equivalent normalized version of the operator which instead acts by $f(z) \mapsto z^{-1} T_g f(z)$. The two excellent survey articles [2] and [14] explain how this class of operators appears in several problems of complex analysis and operator theory.

For a given space of analytic functions X, some relevant and often studied questions concerning the generalized Cesàro operators are listed below.

(i) For what symbols g is the operator $T_g : X \to X$ bounded?

(ii) More generally, for what symbols g does the operator T_g belong to some specific class of bounded operators, e.g. compact?

(iii) If T_g is bounded, can the spectrum of T_g on X be satisfactorily characterized? In particular, how does the spectrum depend on properties of the symbol g?

Answers to the above questions have been obtained for some classical spaces of analytic functions, and references to some of those works are available in the introductory section of [**Paper I**]. My work in that paper is concerned with the action of T_g on the class of so-called growth spaces $A^{-\alpha}, \alpha > 0$, which is a family of Banach spaces each of which consists of functions $f : \mathbb{D} \to \mathbb{C}$ that satisfy

$$\|f\|_{-\alpha} := \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f(z)| < \infty.$$

The class appears notably in the solutions to several famous problems of the theory of Bergman spaces. Questions (i) and (ii) above are rather easily settled for the class of growth spaces and the answers have been previously known. The boundedness and compactness of the operator T_g on $A^{-\alpha}$ is independent of $\alpha > 0$, and the corresponding conditions are that g is contained in the Bloch space $\mathcal{B}$ for boundedness, and in the little Bloch space $\mathcal{B}_0$ for compactness. The Bloch space $\mathcal{B}$ consists of functions satisfying

$$\|g\|_{\mathcal{B}} := \sup_{z \in \mathbb{D}} (1 - |z|^2) |g'(z)| < \infty,$$

while the little Bloch space $\mathcal{B}_0$ is the subspace consisting of functions satisfying

$$\lim_{r\to 1} \sup_{r<|z|<1} (1-|z|^2)|g'(z)| = 0.$$

Thus [**Paper I**] is mainly exploring the last of the questions above. The method employed originates in a very clever idea of Alexandru Aleman and Olivia Constantin from [3], where they study the analogous problem of characterizing the spectrum of T_g acting on the Bergman spaces by translating it into the task of characterizing positive weight functions $w : \mathbb{D} \to \mathbb{R}^+$ with certain properties. In the case of growth spaces, an easy argument based on their idea shows that a non-zero complex number $\lambda \in \mathbb{C}$ belongs to the complement of the spectrum of T_g acting on $A^{-\alpha}$ if and only if the weight

$$w(z) = \left|e^{\frac{g(z)}{\lambda}}\right|(1-|z|^2)^\alpha. \tag{1}$$

has the property that for analytic functions f we have

$$\sup_{z\in\mathbb{D}} w(z)|f(z)| \sim \sup_{z\in\mathbb{D}} w(z)(1-|z|^2)|f'(z)| + |f(0)|. \tag{2}$$

The meaning of $\sim$ here is that the two quantities are comparable, independently of f. Such a restatement of the problem facilitates the use of tools of real and complex analysis. Main part of [**Paper I**] is devoted to establishing a sufficiently useful characterization of weights w satisfying (2). Being equipped with such a weight characterization, and therefore a spectrum characterization, some interesting facts on the behaviour of the spectra of T_g operators acting on growth spaces can be derived. The rest of [**Paper I**] is devoted to this task.

Building on similar ideas and similar spectrum characterizations of T_g acting on the Hardy spaces [5] and the Bergman spaces [3], we take on the task to extend some of the results on growth spaces from [**Paper I**] to a wider range of spaces in [**Paper II**]. We also explore some completely new directions, and we find connections between the generalized Cesàro operators and certain approximation problems.

2.2 Main results The main result of [**Paper I**] characterizes the spectrum $\sigma(T_g|A^{-\alpha})$ of T_g acting on $A^{-\alpha}$. In the statement below $\rho(T_g|A^{-\alpha})$ is the resolvent set, i.e., the complement of the spectrum.

Theorem A (Theorem 5.3 in [**Paper I**]). *Assume that $g \in \mathcal{B}, \lambda \in \mathbb{C} \setminus \{0\}$ and let*

$$w(z) = \left|e^{\frac{g(z)}{\lambda}}\right|(1-|z|^2)^\alpha.$$

The following are equivalent:

(i) $\lambda \in \rho(T_g|A^{-\alpha})$.

(ii) For some $\delta > -1$, the weight w satisfies

$$\sup_{z\in\mathbb{D}} w(z) \int_{\mathbb{D}} \frac{1}{w(\zeta)} \frac{(1-|\zeta|^2)^\delta}{|1-z\bar{\zeta}|^{\delta+2}}\, dA(\zeta) < \infty.$$

For weights of the form (1), the condition (2) is thus equivalent to the conditions stated in the theorem. The condition (ii) allows for explicit description of the spectrum for certain classes of symbols, for instance whenever g' is a rational function.

The characterization of Theorem A above can be used to derive an interesting spectral stability result. In fact, this same stability is present in Hardy and standard weighted Bergman spaces, as is shown in [**Paper II**]. See the paper [**Paper II**] for precise definitions of the spaces mentioned.

Theorem B (Theorem 5.4 in [**Paper I**], Theorem 2.1 in [**Paper II**]). *Let X be one the growth spaces $A^{-\alpha}$ or, for $p \in (0,\infty)$, one of the Hardy spaces H^p or Bergman spaces $L^{p,\alpha}_a$. Let g, h induce bounded operators T_g and T_h on X, and further assume that $\sigma(T_h|X) = \{0\}$. Then*

$$\sigma(T_{g+h}|X) = \sigma(T_g|X).$$

The result I think is interesting in its own right, and also makes it possible to find the spectrum for a large class of symbols which arise as such perturbations of symbols with known spectra, such as symbols with rational derivative. For instance, in the case of growth spaces or the Bergman spaces, perturbations by bounded analytic functions in H^∞ and functions in $\mathcal{B}_0$ do not change the spectrum, since symbols h belonging to these spaces satisfy the condition of the above theorem. In the case of Hardy spaces, the boundedness of the operator T_g is equivalent to $g \in$ **BMOA**, the space of functions of bounded mean oscillation, and it is known that $\sigma(T_h|H^p) = \{0\}$ whenever $h \in$ **VMOA**, the space of functions of vanishing mean oscillation. Our next result identifies an even larger class of symbols for which the corresponding operator spectrum consists of point 0 alone.

Theorem C (Theorem 2.2 and Corollary 2.3 of [**Paper II**]). *If g lies in the norm-closure of H^∞ in BMOA or in the norm-closure of H^∞ in $\mathcal{B}$, then we have that $\sigma(T_g|H^p) = \{0\}$ and that $\sigma(T_g|L^{p,\alpha}_a) = \sigma(T_g|A^{-\alpha}) = \{0\}$, respectively.*

Theorem C of course extends the applicability of Theorem B, but it also suggests the (perhaps naive at first sight) question of what can be said about the converse statement. If the spectrum of T_g acting on some Hardy/Bergman/growth space consists of the point zero alone, does then g lie in the norm-closure of H^∞ in **BMOA**/$\mathcal{B}$? As it turns out, this question has a positive answer in the case of Hardy spaces and in the other cases it is related to a long-standing open problem. For Hardy spaces, we established a slightly stronger result which also sheds some light on the geometric structure of the spectrum.

Theorem D (Theorem 2.4 of [**Paper II**]). *If $g \in$ BMOA is such that for some $0 < p < \infty$ the spectrum $\sigma(T_g|H^p)$ does not contain non-zero points on the real or imaginary axes, then g lies in the norm-closure of H^∞ in BMOA.*

In conjunction with Theorem C we easily see that in the case of Hardy spaces whenever there exist two orthogonal lines through the origin which do not intersect the spectrum in more than the point 0, then actually the spectrum consists of this point alone.

2.3 Directions for further work

The most intriguing question that has been left unanswered is if there exists a Bergman/growth space version of Theorem D. Unfortunately, our proof of Theorem D uses techniques which are very exclusive to the Hardy space setting, and it cannot be adapted to a Bergman-like setting.

Problem. *Can a function $g \in \mathcal{B}$ be approximated in the $\mathcal{B}$-norm by a bounded function, given that the spectrum of T_g on a Bergman space or a growth space equals $\{0\}$?*

The question of characterizing the closure of H^∞ in $\mathcal{B}$ has been stated in [7] and remains open to this date. Note that if the problem stated above has an affirmative solution, then a rather satisfying characterization of the closure of H^∞ in $\mathcal{B}$ would be obtained in terms of the condition (ii) of Theorem A, or in terms of the corresponding condition for the Bergman spaces. That is, in the case of growth spaces, $g \in \mathcal{B}$ would lie in the closure of H^∞ if and only if the weights $w(z) = w_\lambda(z) = \left|e^{\frac{g(z)}{\lambda}}\right|(1-|z|^2)^\alpha$ satisfy condition (ii) of Theorem A for all non-zero $\lambda \in \mathbb{C}$ (see [3] and [6], or [**Paper II**] for the exact condition characterizing the spectrum of T_g on the Bergman spaces).

3 Hilbert spaces of analytic functions with a contractive backward shift

3.1 Background

The results of the second project are contained in the papers [**Paper III**] and [**Paper IV**], both written in collaboration with my advisor Alexandru Aleman. The topic of our study is the so-called backward shift operator, denoted here by L, which acts on analytic functions $f : \mathbb{D} \to \mathbb{C}$ by the formula

$$Lf(z) = \frac{f(z) - f(0)}{z}.$$

The backward shift is of major importance in operator theory for its operator modelling properties. Under some very natural assumptions, for any operator $T : X \to X$ on a Hilbert space X with operator norm bounded by 1, there exists a Hilbert space of analytic functions $\mathcal{H}$ such that T is unitarily equivalent to L acting on $\mathcal{H}$. In general the modelling space $\mathcal{H}$ will consist of vector-valued functions. This result is a consequence of operator model theories of de Branges-Rovnyak and of Sz.-Nagy-Foias (see [8] and [15]).

The setting for our study is a general Hilbert space $\mathcal{H}$ of analytic functions which satisfies the following properties:

(A.1) the evaluation $f \mapsto f(\lambda)$ is a bounded linear functional on $\mathcal{H}$ for each $\lambda \in \mathbb{D}$, (A.2) $\mathcal{H}$ is invariant under the backward shift operator L and

$$\|Lf\|_{\mathcal{H}} \leq \|f\|_{\mathcal{H}}, \quad f \in \mathcal{H},$$

(A.3) the constant function 1 is contained in $\mathcal{H}$ and has the reproducing property

$$\langle f, 1 \rangle_{\mathcal{H}} = f(0), \quad f \in \mathcal{H}.$$

The class of spaces satisfying the above very general assumptions is vast. Up to inessential normalization to make (A.3) hold, it includes not only the classical operator model spaces such as $K_\theta := H^2 \ominus \theta H^2$ for θ inner function, and de Branges-Rovnyak spaces $\mathcal{H}(b)$, but also classes of spaces usually not associated with model theories, such as for instance Dirichlet-type spaces. Again, for precise definitions please see the corresponding paper [**Paper IV**].

The motivation for our research is the observation that the above three assumptions imply some non-trivial structural properties of the spaces in question, and allow for computation of the norms in the space in a very special way. This is the content of the rather technical Theorem 2.2 of [**Paper IV**], the precise statement of which I choose to omit in this exposition. In essence however, to any space $\mathcal{H}$ in our class there exists an associated analytic row operator $\mathbf{B} := (b_i)_{i=1}^{\infty}$ such that the reproducing kernel $k_{\mathcal{H}}$ of $\mathcal{H}$ can be expressed as

$$k_{\mathcal{H}}(z, \lambda) = \frac{1 - \sum_{i \geq 1} \overline{b_i(\lambda)} b_i(z)}{1 - \overline{\lambda} z} = \frac{1 - \mathbf{B}(z)\mathbf{B}(\lambda)^*}{1 - \overline{\lambda} z} \tag{3}$$

We will denote $\mathcal{H}$ by $\mathcal{H}[\mathbf{B}]$ if the kernel is given by (3). Based on the structure of the kernel, we construct a special isometric embedding operator $J : \mathcal{H} \to H^2 \oplus \Delta L^2$ of the form $Jf = (f, \mathbf{g})$, where $\Delta = \mathbf{B}^*\mathbf{B}$ and L^2 is the space of square summable functions on the unit circle $\mathbb{T}$ which take values in some vector space. The orthogonal complement of the image of this embedding is nicely characterizable in terms of $\mathbf{B}$, and certain useful intertwining relations between J and L are present. Most of the main results of the research are derived as a consequence of the existence of this embedding.

3.2 Main results Consequences of the research include, among other results, a solution to an open problem on approximation in de Branges-Rovnyak spaces stated in [9] and a surprising and broad generalization, answers to questions regarding reverse Carleson embeddings from [10] together with further development of the theory of these embeddings, and the development of the theory of a natural generalization of scalar-valued de Branges-Rovnyak spaces.

3.2.1 Continuous approximation. In [9] Fricain discusses the problem of norm approximation of general functions in de Branges-Rovnyak spaces $\mathcal{H}(b)$ by functions in the disk algebra $\mathcal{A}$, the algebra of analytic functions in $\mathbb{D}$ with continuous extensions to the closure of the disk clos($\mathbb{D}$). The spaces $\mathcal{H}(b)$ are precisely those Hilbert spaces of analytic functions which admit a reproducing kernel of the form given in (3) with $\mathbf{B} = (b, 0, 0, \ldots)$. The approximation has been known to be possible in two important special cases. The first case, when $\mathcal{H}(b)$ is contained isometrically inside the Hardy space H^2, is the case when b is an inner function. The density of continuous functions in this setting has been confirmed by Aleksandrov in [1]. The second case is when $\log(1 - |b|)$ is integrable with respect to the Lebesgue measure on the unit circle, that is, in the case of the so-called non-extreme de Branges-Rovnyak space. Sarason proved that then the analytic polynomials are contained in $\mathcal{H}(b)$, and that they moreover form a dense subset of the space (see, for instance, [13]).

The main result of [**Paper III**] solves in the affirmative the approximation problem for de Branges-Rovnyak spaces. The intersection $\mathcal{H}(b) \cap \mathcal{A}$ is always norm-dense in $\mathcal{H}(b)$. However, in [**Paper IV**] we have obtained a much more striking conclusion which includes the result of [**Paper III**] as a special case. The next theorem is in my view very unexpected. It is used throughout [**Paper IV**] in technical arguments, but should be interesting on its own.

Theorem E (Theorem 3.5 in [**Paper IV**]). *Let $\mathcal{H}$ be a Hilbert space of analytic functions which satisfies the properties (A.1)-(A.3) above. Then the intersection $\mathcal{H} \cap \mathcal{A}$ is norm-dense in $\mathcal{H}$.*

In fact, a similar result holds in the case when the functions in $\mathcal{H}$ take values in a finite dimensional Hilbert space. The function with continuous extensions to the closure of $\mathbb{D}$ are norm-dense in spaces satisfying natural versions of properties (A.1)-(A.3) in the vector-valued setting (see Section 2 of [**Paper IV**] for details).

3.2.2 Reverse Carleson measures. Theorem E facilitates the study of forward and reverse Carleson measures for the class of spaces considered here, and this was indeed one of the principal reasons for proving it in the special case $\mathcal{H} = \mathcal{H}(b)$ in [**Paper III**]. Our results pertain to the reverse case. A finite Borel measure on clos($\mathbb{D}$) is a reverse Carleson measure for $\mathcal{H}$ if there exists a constant $C > 0$ such that the estimate $\|f\|^2_{\mathcal{H}} \leq C \int_{\text{clos}(\mathbb{D})} |f(z)|^2 d\mu(z)$ holds at least for functions f belonging to some dense subset of $\mathcal{H}$. Note that the integral on the right-hand side does not make sense for arbitrary analytic functions f in $\mathbb{D}$, since the measure μ might contain the boundary of $\mathbb{D}$ in the support. However, for the spaces satisfying our assumptions, Theorem E provides us with a dense set of functions for which the integral is well-defined and on which the condition above can be tested.

We proved two theorems in the context of reverse Carleson measures. The first one should be compared to results established in [10, Theorem 2.4] which deals with the special case $\mathcal{H} = \mathcal{H}(b)$ for non-extreme b. In that case, the space $\mathcal{H}(b)$ is invariant for the operator M_z of multiplication by z: $f(z) \mapsto zf(z)$.

Theorem F (Theorem 4.2 of [**Paper IV**]). *Let $\mathcal{H}$ be a Hilbert space of analytic functions which satisfies the properties (A.1)-(A.3) above, and moreover is invariant for M_z. Then the following are equivalent.*

(i) $\mathcal{H}$ admits a reverse Carleson measure.

(ii)

$$\sup_{0<r<1} \int_{\mathbb{T}} \left\| \frac{\sqrt{1-|r\lambda|^2}}{1-r\overline{\lambda}z} \right\|_{\mathcal{H}}^2 dm(\lambda) < \infty.$$

(iii) If k is the reproducing kernel of $\mathcal{H}$, then

$$\sup_{0<r<1} \int_{\mathbb{T}} \frac{1}{(1-|r\lambda|^2)k(r\lambda, r\lambda)} dm(\lambda) < \infty.$$

If the above conditions are satisfied, then the measures $h_1 dm, h_2 dm$ on the circle given by

$$h_1(\lambda) := \lim_{r\to 1} \left\| \frac{\sqrt{1-|r\lambda|^2}}{1-r\overline{\lambda}z} \right\|_{\mathcal{H}}^2$$

and

$$h_2(\lambda) := \lim_{r\to 1} \frac{1}{(1-|r\lambda|^2)k(r\lambda, r\lambda)}$$

define reverse Carleson measures for $\mathcal{H}$. Moreover, if ν is any reverse Carleson measure for $\mathcal{H}$ and v is the density of the absolutely continuous part of the restriction of ν to $\mathbb{T}$, then $h_1 dm$ and $h_2 dm$ have the following minimality property: there exist constants $C_i > 0, i = 1, 2$ such that

$$h_i(\lambda) \leq C_i v(\lambda)$$

for almost every $\lambda \in \mathbb{T}$.

We also obtain a negative result on existence of reverse Carleson measures for spaces satisfying a norm equality related to L.

Theorem G (Theorem 4.4 of [**Paper IV**]). *Let $\mathcal{H}$ be a Hilbert space of analytic functions which satisfies the properties (A.1)-(A.3) above, and moreover that the identity*

$$\|Lf\|_{\mathcal{H}}^2 = \|f\|_{\mathcal{H}}^2 - |f(0)|^2$$

holds in $\mathcal{H}$. If $\mathcal{H}$ admits a reverse Carleson measure, then $\mathcal{H}$ is isometrically contained in the Hardy space H^2.

Examples of spaces satisfying the norm condition include $\mathcal{H}(b)$ for extreme b. Consequently the above theorem answers a question in [10] whether such a space admits a reverse Carleson measure. Our result implies that this happens only in the case when b is an inner function.

3.2.3 Finite rank spaces. A generalization of $\mathcal{H}(b)$-spaces is obtained by considering reproducing kernels of the form (3) where only finitely many terms b_i are non-zero. Thus, when $\mathbf{B} = (b_1, \dots, b_n, 0, 0 \dots)$ and the reproducing kernel of $\mathcal{H}$ has the form

$$k_{\mathcal{H}}(z, \lambda) = \frac{1 - \sum_{i=1}^{n} \overline{b_i(\lambda)} b_i(z)}{1 - \overline{\lambda} z}.$$

In [**Paper IV**] we call such spaces *finite rank spaces*. We studied mainly the case when the space is invariant under the operator M_z and it turns out that under this assumption the spaces have much in common with the classical non-extreme $\mathcal{H}(b)$. For instance, for each such space there exists an n-by-n matrix-valued analytic function **A** which is a kind of Pythagorean mate of **B** in the sense that $\mathbf{B}^*\mathbf{B} + \mathbf{A}^*\mathbf{A} = I$ on the circle $\mathbb{T}$, where I is the identity matrix. An analytic function $f \in H^2$ is contained in $\mathcal{H}[\mathbf{B}]$ if and only if there exists a $\mathbb{C}^n$-valued analytic function **g** in the $\mathbb{C}^n$-valued Hardy space $H^2(\mathbb{C}^n)$ such that

$$P_+ \mathbf{B}^* f = P_+ \mathbf{A}^* \mathbf{g},$$

where P_+ is the component-wise orthogonal projection from L^2 of the circle onto H^2. If such a **g** exists, then we have that $\|f\|^2_{\mathcal{H}[\mathbf{B}]} = \|f\|^2_{H^2} + \|\mathbf{g}\|^2_{H^2}$. In fact, our isometric embedding J mentioned earlier takes f into the tuple $(f, \mathbf{g})$.

We prove some generalizations of classical theorems for $\mathcal{H}(b)$-spaces, including the density of polynomials and the structure of backward shift invariant subspaces, all with new proofs which utilize the embedding J. Moreover, we obtain the following result which sheds some light on the structure of M_z-invariant subspaces. The result is new even for $\mathcal{H}(b)$.

Theorem H (Theorem 5.11 of [**Paper IV**]). *Let $\mathcal{H} = \mathcal{H}[\mathbf{B}]$ be of finite rank, M_z-invariant and $\mathcal{M}$ be a closed M_z-invariant subspace of $\mathcal{H}$. Then*

(i) $\dim \mathcal{M} \ominus M_z \mathcal{M} = 1$,

(ii) any non-zero element in $\mathcal{M} \ominus M_z\mathcal{M}$ is a cyclic vector for $M_z|\mathcal{M}$,

(iii) if $\phi \in \mathcal{M} \ominus M_z\mathcal{M}$ is of norm 1, then there exists a space $\mathcal{H}[C]$ invariant under M_z, where $C = (c_1, \dots, c_k)$ and $k \leq n$, such that

$$\mathcal{M} = \phi \mathcal{H}[C]$$

and the mapping $g \mapsto \phi g$ is an isometry from $\mathcal{H}[C]$ onto $\mathcal{M}$,

(iv) if $\phi \in \mathcal{M} \ominus M_z\mathcal{M}$ with $J\phi = (\phi, \boldsymbol{\phi}_1)$, then

$$\mathcal{M} = \left\{ f \in \mathcal{H}[B] : \tfrac{f}{\phi} \in H^2, \tfrac{f}{\phi} \boldsymbol{\phi}_{\mathbf{1}} \in H^2(\mathbb{C}^n) \right\}.$$

3.3 Directions for further work

The proof of Theorem E in the finite dimensional vector-valued setting is based on a duality argument which uses the characterization of the dual space of the disk algebra $\mathcal{A}$ as the space of Cauchy transforms. For analytic functions f with continuous extensions to clos($\mathbb{D}$) which take values in an infinite dimensional vector space a similarly nice description does not seem to be available. Consequently, the following question remains open.

Problem. *Assume that $\mathcal{H}$ consists of functions taking values in an infinite dimensional vector space, and that $\mathcal{H}$ satisfies the vector-valued analogues of properties (A.1)-(A.3). Are the functions with continuous extensions to clos($\mathbb{D}$) dense in the space?*

There is also more to be discovered about the structure of M_z-invariant subspaces. The conclusion (i) of Theorem H does not hold in the case when the hypothesis of finite rank is dropped. This follows from the work of Esterle [11], who indeed showed that $\dim \mathcal{M} \ominus M_z\mathcal{M}$ can be infinite, even when M_z is unitarily equivalent to a deceptively simple-looking weighted shift operator. The classical Dirichlet space is an example of an infinite rank space for which both the conclusion (i) and (ii) of Theorem H hold. This is a result of Stefan Richter found in [12]. Unlike the finite rank case, we have been unable to reprove his result using our model and the embedding J. It is of interest to me if these results are obtainable through these means.

An even more ambitious task is the following. Let $\mathcal{H}$ be a space of infinite rank which is M_z-invariant and in which the polynomials are dense, and let $\mathcal{M}$ be a subspace for which conclusion (i) of Theorem H is satisfied. Is then $\phi \in \mathcal{M} \ominus M_z\mathcal{M}$ a cyclic vector for M_z acting on $\mathcal{M}$? The question appears to be very difficult, and the following equivalent formulation can be deduced from our results in [**Paper IV**].

Problem. *Let $\mathcal{H}$ be a Hilbert space satisfying (A.1)-(A.3) above, invariant for M_z and such that the sequence of powers of the backward shift $\{L^n\}_{n=1}^{\infty}$ converge to zero in the strong operator topology on $\mathcal{H}$. Are the polynomials then dense in $\mathcal{H}$?*

To me personally, an explicit counterexample would be (nearly) as exciting as an affirmative solution.

4 Nearly invariant subspaces of de Branges spaces

4.1 Background

[**Paper V**] is concerned with the concept of nearly invariance. This notion appears also in the paper [**Paper IV**] in the context of function spaces on the unit disk, while here we work instead in certain spaces of entire functions called de Branges spaces.

A space of analytic functions is said to be nearly invariant if zeros of functions in the space can be divided out without leaving the space. More precisely, a space $\mathcal{H}$ is nearly invariant if for any $f \in \mathcal{H}$, the function $f(z)/(z-\lambda)$ is in $\mathcal{H}$ for any $\lambda \in \mathbb{C}$ such that

$f(\lambda) = 0$. Of course, in the case that the functions in $\mathcal{H}$ all have a zero in common at some point λ, then this zero cannot be divided out without leaving the space. Therefore, more generally, we say that a space is nearly invariant if all zeros which are not common zeros of the functions in the space can be divided out in the way indicated above.

The de Branges spaces $\mathcal{H}(E)$ form a family of Hilbert spaces of entire functions, parametrized by entire functions E which satisfy the inequality $|E(z)| > |E(\overline{z})|$ for z in the upper half plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. To define the de Branges space $\mathcal{H}(E)$ associated to E, recall the standard Hardy space of the upper half plane $H^2(\mathbb{C}^+)$, and for entire functions let us define the flip operation $f^*(z) := \overline{f(\overline{z})}$. Then $\mathcal{H}(E)$ is the Hilbert space of entire functions f which satisfy the following three properties:

(i) $f/E \in H^2(\mathbb{C}^+)$,

(ii) $f^*/E \in H^2(\mathbb{C}^+)$,

(iii) $\|f\|^2_{\mathcal{H}(E)} := \int_{\mathbb{R}} |f/E|^2 dx < \infty$.

The norm of the space is the one indicated in (iii). The de Branges spaces are reproducing kernel Hilbert spaces with kernels of the form

$$k_E(\lambda, z) = \frac{E(z)\overline{E(\lambda)} - E^*(z)\overline{E^*(\lambda)}}{2\pi i(\overline{\lambda} - z)}.$$

Perhaps the most recognized Hilbert spaces of analytic functions which are de Branges spaces are the Paley-Wiener spaces PW_a, which consist of entire functions that are the Fourier transforms of measurable functions in $L^2(-a, a)$, and correspond to $E(z) = \exp(-iaz)$.

4.2 Main results The result of [**Paper V**] is the following description of nearly invariant subspaces of de Branges spaces.

Theorem I (Theorem 1.1 in [**Paper V**]). *Let $\mathcal{N}$ be a nearly invariant subspace with no common zeros of a de Branges space $\mathcal{H}(E)$. Then there exists a de Branges space $\mathcal{H}(E_0)$ and $\alpha \in \mathbb{R}$ such that*

$$\mathcal{N} = e^{i\alpha z}\mathcal{H}(E_0) = \{e^{i\alpha z} f(z) : f \in \mathcal{H}(E_0)\}.$$

This result is used to obtain a simple proof of a more precise description from [4] in the particular case that $\mathcal{H}(E) = PW_a$. Namely, if $\mathcal{N}$ is a nearly invariant subspace of PW_a with no common zeros, then there exists an interval $I \subseteq (-a, a)$ such that

$$\mathcal{N} = \{f \in PW_a : \operatorname{supp} \hat{f} \subset I\}.$$

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Paper I

Spectra of generalized Cesàro operators acting on growth spaces

Bartosz Malman

Abstract

We study the spectrum of generalized Cesàro operators T_g acting on the class of growth spaces $A^{-\alpha}$. We show how the problem of determining the spectrum is related to boundedness of standard weighted Bergman projections on weighted L^∞-spaces. Using this relation we establish some general spectral properties of these operators, and explicitly compute the spectrum for a large class of symbols g.

1 Introduction and statements of the main results

Let $\mathbb{D}$ denote the open unit disk of the complex plane $\mathbb{C}$. For a fixed analytic function $g : \mathbb{D} \to \mathbb{C}$ with $g(0) = 0$ we define the corresponding generalized Cesàro operator T_g acting on an analytic function $f : \mathbb{D} \to \mathbb{C}$ by the formula

$$T_g f(z) = \int_0^z g'(\zeta) f(\zeta)\, d\zeta.$$

For the particular choice of $g(z) = \log(\frac{1}{1-z})$ we obtain the classical (shifted) Cesàro operator, which can alternatively be defined by its action on the Taylor coefficients of $f(z) = \sum_{n=0}^\infty a_n z^n$ by the formula

$$\sum_{n=0}^\infty a_n z^n \mapsto \sum_{n=0}^\infty b_n z^{n+1},$$

where

$$b_n = \frac{1}{n+1} \sum_{k=0}^n a_k.$$

The action of generalized Cesàro operators on classical spaces of analytic functions has been studied in a number of articles. It was noted in [14] that T_g is bounded on the Hardy space H^2 if and only if g is a function of bounded mean oscillation, and this fact was used to obtain a short proof of the analytic John-Nirenberg inequality. Further results on boundedness and compactness of T_g acting on the Hardy spaces H^p for $p \neq 2$ have been obtained in [5], and a characterization of the spectrum of T_g acting on those spaces is available in [3]. Further properties of the spectrum of T_g acting on H^2 are described in [19]. The corresponding questions have also been studied in the context of the Bergman spaces (see [6] and [2]). More recently, T_g operators with entire symbols $g : \mathbb{C} \to \mathbb{C}$ acting on spaces of entire functions have been studied in [10], [11] and [8]. A study of the spectrum of a certain normalized version of T_g acting on a class of Banach spaces of analytic function satisfying some natural assumptions is carried out in [4]. The two excellent survey articles [1] and [16] contain more references and mention how this class of operators appear in other parts of analysis.

The purpose of this article is to study the spectrum of T_g acting on the so-called *growth spaces*. For $\alpha > 0$, the growth space $A^{-\alpha}$ is the Banach space of analytic functions defined in $\mathbb{D}$ for which the quantity

$$\|f\|_{-\alpha} := \sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha}|f(z)|$$

is finite. Growth spaces appear in several contexts, perhaps most importantly in the study of zero sequences and interpolating sequences for the classical Bergman spaces (see [13, Chapter 4 and 5]). The closure of analytic polynomials in the norm $\|\cdot\|_{-\alpha}$ is denoted by $A_0^{-\alpha}$ and consists precisely of those functions $f \in A^{-\alpha}$ for which

$$\lim_{r\to 1} \sup_{r<|z|<1} (1-|z|^2)^{\alpha}|f(z)| = 0.$$

The space $A_0^{-\alpha}$ has the big advantage of being separable, and it turns out that T_g exhibits largely the same behaviour on $A^{-\alpha}$ and $A_0^{-\alpha}$. As we will see in Section 3, T_g is simultaneously bounded or compact on both spaces, and the spectrum of T_g is the same for both spaces. The boundedness and compactness questions have already been studied in [17], where it is shown that T_g is bounded on $A^{-\alpha}$ if and only if g satisfies

$$\|g\|_{\mathcal{B}} := \sup_{z\in\mathbb{D}} (1-|z|^2)|g'(z)| < \infty,$$

that is, if and only if g is contained in the Bloch space $\mathcal{B}$. The operator is compact if and only if $g \in \mathcal{B}_0$, the little Bloch space, which is the subspace of $\mathcal{B}$ consisting of functions which satisfy

$$\lim_{r\to 1} \sup_{r<|z|<1} (1-|z|^2)|g'(z)| = 0.$$

Our study of the spectrum $\sigma(T_g|A^{-\alpha})$ follows an idea of [2] which translates the spectral problem into an equivalent problem of characterizing weights $w : \mathbb{D} \to (0, \infty)$ with a certain property. It starts with the observation that for $\lambda \in \mathbb{C} \setminus \{0\}$ the unique analytic solution h to the equation

$$(1 - \lambda^{-1}T_g)h = f$$

is given by

$$h(z) = R_{\lambda,g} f(z) := f(0) e^{\frac{g(z)}{\lambda}} + e^{\frac{g(z)}{\lambda}} \int_0^z e^{-\frac{g(\zeta)}{\lambda}} f'(\zeta)\, d\zeta.$$

It is easy to see that the operator $T_g - \lambda$ is injective, and hence it will be invertible on any Banach space X if and only if the operator $R_{\lambda,g}$ acts boundedly on X. In the case $X = A^{-\alpha}$, this in turn implies an equivalence of norms on a certain Banach space of analytic functions which we will now describe. To this end, let w be a weight on $\mathbb{D}$. For us, this will mean a strictly positive continuous function $w : \mathbb{D} \to (0, \infty)$. Consider the space L^∞_w consisting of (equivalence classes of) measurable functions f defined in $\mathbb{D}$ which satisfy $\|wf\|_\infty < \infty$, where $\|\cdot\|_\infty$ denotes the usual essential supremum norm. Let H^∞_w be the closed subspace of L^∞_w consisting of analytic functions. For $\lambda \in \mathbb{C} \setminus \{0\}$ let

$$w(z) = \left|e^{\frac{g(z)}{\lambda}}\right|(1 - |z|^2)^\alpha. \tag{1}$$

It turns out (see Proposition 5.1 below) that λ is in the resolvent set $\rho(T_g|A^{-\alpha})$ if and only if for $f \in H^\infty_w$ we have the equivalence of norms

$$\sup_{z\in\mathbb{D}} w(z)|f(z)| \sim \sup_{z\in\mathbb{D}} w(z)(1 - |z|^2)|f'(z)| + |f(0)|. \tag{2}$$

We are thus lead to studying weights w for which such an equivalence holds. We show in Section 4 that the existence of such a norm equivalence is connected to the boundedness on L^∞_w of the weighted Bergman projections. For $\delta > -1$, the weighted Bergman projection P_δ is given by

$$P_\delta f(z) = (\delta + 1) \int_{\mathbb{D}} f(\zeta) \frac{(1 - |\zeta|^2)^\delta}{(1 - z\bar{\zeta})^{2+\delta}} dA(\zeta),$$

where dA denotes the area measure on $\mathbb{D}$. For the class of bounded and differentiable weights satisfying for some constant $k_w > 0$ the inequality

$$(1 - |z|)|\nabla w(z)| \le k_w w(z) \tag{3}$$

we prove the following theorem (see Section 4 for the definition of $\tilde{P}_\delta$ appearing below).

Theorem 4.5. *Let $w : \mathbb{D} \to (0, \infty)$ be a bounded and differentiable weight which satisfies* (3). *The following are equivalent:*

(i) The operator P_δ is bounded on L^∞_w for some $\delta > -1$.

(ii) The operator $\tilde{P}_\delta$ is bounded on L^∞_w for some $\delta > -1$

(iii) There exists $\delta > -1$ such that

$$\sup_{z\in\mathbb{D}} w(z) \int_{\mathbb{D}} \frac{1}{w(\zeta)} \frac{(1 - |\zeta|^2)^\delta}{|1 - z\bar{\zeta}|^{\delta+2}} dA(\zeta) < \infty.$$

(iv) For each $n \geq 1$

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^n w(z)|f^{(n)}(z)| + \sum_{k=0}^{n-1} |f^{(k)}(0)|$$

defines an equivalent norm on H_w^∞.

The first three conditions of the above theorem are equivalent, with the same constant δ, even without the assumption that (3) holds (see Theorem 4.1). The fourth condition is on the other hand crucial for our further study of the spectrum of T_g on the growth spaces. We note that the general problem of boundedness of weighted Bergman projections on L_w^∞ has been studied in [9].

Results of Section 4 are applied to the spectral problem in Section 5. The main difficulty is a technical one of extending (2) to higher derivatives, i.e., to establish that condition (iv) of Theorem 4.5 holds for the weight given by (1) whenever $\lambda \in \rho(T_g|A^{-\alpha})$. The main tool here is an operator interpolation theorem which we prove in Section 2 (see Theorem 2.2 below), and the result is the following characterization of the spectrum of T_g acting on $A^{-\alpha}$.

Theorem 5.3. *Let $g \in \mathcal{B}$, $\lambda \in \mathbb{C} \setminus \{0\}$ and*

$$w(z) = \left|e^{\frac{g(z)}{\lambda}}\right|(1-|z|^2)^\alpha.$$

The following are equivalent:

(i) $\lambda \in \rho(T_g|A^{-\alpha})$.

(ii) For some $\delta > -1$, the weight w satisfies

$$\sup_{z\in\mathbb{D}} w(z) \int_{\mathbb{D}} \frac{1}{w(\zeta)} \frac{(1-|\zeta|^2)^\delta}{|1-z\overline{\zeta}|^{\delta+2}}\, dA(\zeta) < \infty.$$

This characterization allows us to explicitly compute the spectrum for a class of symbols $g \in \mathcal{B}$. It turns out, precisely as is the case for many other Banach spaces X, that the spectrum of T_g acting on $X = A^{-\alpha}$ satisfies the equality

$$\sigma(T_g|X) = \{0\} \bigcup \overline{\left\{\lambda \in \mathbb{C}\setminus\{0\} : e^{g/\lambda} \notin X\right\}} \tag{4}$$

whenever g is the primitive of a rational function. Moreover, we obtain the following spectral stability result.

Theorem 5.4. *Let $g, h \in \mathcal{B}$ and assume that $\sigma(T_h|A^{-\alpha}) = \{0\}$. Then*

$$\sigma(T_{g+h}|A^{-\alpha}) = \sigma(T_g|A^{-\alpha}).$$

This allows us to extend (4) to the case when g is the sum of a primitive of a rational function with a bounded analytic function and a function in $\mathcal{B}_0$ (see Theorem 5.6).

For a general Banach space X and a general symbol g, the equality in (4) does not hold. This is shown in [2] in the case when X is any of the standard Bergman spaces. By relating condition (ii) of Theorem 5.3 to the classical Békollé weight condition, we show that the example in [2] can be adapted to the case considered here, so that the spectrum of T_g on $A^{-\alpha}$ can indeed be much larger than (4).

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2 Preliminaries

2.1 An integral kernel estimate The following well-known estimate will be used frequently below.

Proposition 2.1. *For any $\delta > -1$ and any $s > 0$ there exists a constant $C > 0$ such that*

$$\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^\delta}{|1-z\overline{\zeta}|^{\delta+2+s}} dA(\zeta) \leq \frac{C}{(1-|z|)^s}.$$

For a proof, see [13, Theorem 1.7].

2.2 Equivalent norms For any $n \geq 0$, the space $A^{-\alpha}$ can equivalently be normed by

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha+n} |f^{(n)}(z)| + \sum_{k=0}^{n-1} |f^{(k)}(0)|$$

(see Theorem 5.5 in [12]). We will often use these norms without further reference.

2.3 Duality relations Let $L^1(\mathbb{D})$ be the Lebesgue space of (equivalence classes of) measurable functions which are integrable with respect to dA. It is a consequence of the results of [15] that the dual space $(A_0^{-\alpha})^*$ can be identified with the quotient space $L^1(\mathbb{D})/N$, where

$$N = \Big\{ f \in L^1(\mathbb{D}) : \int_{\mathbb{D}} f(\zeta)(1-|\zeta|^2)^\alpha h(\zeta) dA(\zeta) = 0, \forall h \in A^{-\alpha} \Big\}.$$

Moreover, it is also proved in [15] that the dual space $(L^1(\mathbb{D})/N)^*$ can be identified with $A^{-\alpha}$. The duality pairing is in both cases given by

$$\int_{\mathbb{D}} f(\zeta)(1-|\zeta|^2) h(\zeta) dA(\zeta),$$

where $f \in L^1(\mathbb{D})/N$ and $h \in A^{-\alpha}$ or $h \in A_0^{-\alpha}$.

It is clear from this characterization that a sequence $\{f_n\}_{n=1}^{\infty}$ in $A_0^{-\alpha}$ converges weakly to f if and only if the norms $\|f_n\|_{-\alpha}$ stay bounded and $f_n(z) \to f(z)$ for each $z \in \mathbb{D}$ as $n \to \infty$, uniformly on compact subsets of $\mathbb{D}$. As the dual space of $L^1(\mathbb{D})/N$, the space $A^{-\alpha}$ can be equipped with the usual weak-* topology, and similarly $\{f_n\}_{n=1}^{\infty}$ in $A^{-\alpha}$ converges weak-* to a function f if and only if the norms $\|f_n\|_{-\alpha}$ stay bounded and $f_n(z) \to f(z)$ uniformly on compact subsets of $\mathbb{D}$. In particular, for $r \in (0,1)$ the dilations $f_r(z) = f(rz)$ of any function $f \in A^{-\alpha}$ converge weak-* to f as $r \to 1$.

2.4 Interpolation of operators on growth spaces In Section 5 we will need the following result on interpolation of linear operators between two growth spaces. The proof consists of applying the basic ideas of the well-known complex interpolation method.

Theorem 2.2. *Let $R : A_0^{-\beta} \to A_0^{-\beta}$ be a bounded linear operator. If for some $\alpha \in (0,\beta)$ the operator R also maps $A_0^{-\alpha}$ boundedly into itself, then the operator R maps $A_0^{-\gamma}$ boundedly into itself for any $\gamma \in (\alpha,\beta)$.*

Proof. Let $C_0(\mathbb{D})$ denote the Banach space of functions continuous in $\overline{\mathbb{D}}$ that vanish on the boundary $\partial\mathbb{D} = \mathbb{T}$, and let

$$S = S(\alpha,\beta) = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}.$$

Fix a function $f \in C_0(\mathbb{D})$. For $z \in \overline{S}$ let

$$h_z(\lambda) = \int_{\mathbb{D}} f(\zeta) \frac{(1-|\zeta|^2)^{\beta-z}}{(1-\lambda\overline{\zeta})^{\beta+2}}\, dA(\zeta), \quad \lambda \in \mathbb{D}$$

Proposition 2.1 implies that $h_z \in A_0^{-\operatorname{Re} z} \subseteq A_0^{-\beta}$ and that the norms $\|h_z\|_{-\beta}$ are bounded uniformly in $z \in \overline{S}$ by a constant multiple of $\|f\|_\infty$, i.e., there exists a constant $C > 0$ such that for all $\lambda \in \mathbb{D}$ and all $z \in \overline{S}$ we have the estimate $|h_z(\lambda)| \le C\|f\|_\infty (1-|\lambda|)^{-\beta}$. In particular, the family $\{h_z\}_{z\in\overline{S}}$ is bounded uniformly on compact subsets of $\mathbb{D}$. It is also clear that $h_{z_n}(\lambda) \to h_z(\lambda)$ uniformly on compact subsets of $\mathbb{D}$ if $z_n \to z \in \overline{S}$, and hence in that case $h_{z_n} \to h_z$ weakly in $A_0^{-\beta}$.

For a fixed $\epsilon > 0$ consider the function $G_\epsilon : \overline{S} \to C_0(\mathbb{D})$ given by

$$G_\epsilon(z) = \exp\left(\frac{-\epsilon}{1-|\cdot|}\right)(1-|\cdot|^2)^z (Rh_z)(\cdot).$$

Note that the exponential factor in the definition of G_ϵ ensures that

$$\sup_{z\in\overline{S}} \|G_\epsilon(z)\|_\infty < \infty.$$

If $z_n \to z \in \overline{S}$, then the above paragraph and the fact that R preserves weak convergence of sequences implies that $Rh_{z_n} \to Rh_z$ weakly in $A_0^{-\beta}$. Then $Rh_{z_n}(\lambda) \to Rh_z(\lambda)$ uniformly on compact subsets of $\mathbb{D}$, and so $\|G_\epsilon(z_n) - G_\epsilon(z)\|_\infty \to 0$ by the rapid decay of $\exp(-\epsilon/(1-|\lambda|))$. This shows continuity of G_ϵ. *We claim that G_ϵ is analytic in S.* We will complete the proof of the theorem under this assumption, and prove analyticity of G_ϵ next. The assumption of boundedness of R on $A_0^{-\alpha}$ together with Proposition 2.1 implies that for all $y \in \mathbb{R}$, we have

$$(1-|\lambda|)^\alpha Rh_{\alpha+iy}(\lambda) \leq C\|f\|_\infty$$

and consequently

$$\|G_\epsilon(\alpha+iy)\|_\infty \leq C\|f\|_\infty,$$

where $C > 0$ depends only on the norm of the operator R on $A_0^{-\alpha}$ but not on $\epsilon > 0$. In the same way we obtain

$$\|G_\epsilon(\beta+iy)\|_\infty \leq C_1\|f\|_\infty,$$

$C_1 > 0$ depending on the norm of R on $A_0^{-\beta}$ but being independent of $\epsilon > 0$. By the vector-valued generalization of the classical Hadamard's three lines theorem we get that $\|G_\epsilon(z)\| \leq C_2\|f\|_\infty$ for all $z \in S$, with a constant $C_2 > 0$ independent of $\epsilon > 0$. Now let $g \in A_0^{-\gamma}$, $\alpha < \gamma < \beta$ and put $f(\zeta) = g(\zeta)(1-|\zeta|^2)^\gamma \in C_0(\mathbb{D})$. In the above notation we have that $h_\gamma = g$, and we conclude that

$$\exp\left(\frac{-\epsilon}{1-|\lambda|}\right)(1-|\lambda|^2)^\gamma|Rg(\lambda)| \leq \|G_\epsilon(\gamma)\|_\infty \leq C_2\|g\|_{-\gamma}.$$

Letting ϵ tend to zero and taking the supremum over $\lambda \in \mathbb{D}$ shows that R is bounded on $A_0^{-\gamma}$.

It remains to prove that $G_\epsilon : S \to C_0(\mathbb{D})$ is analytic. Recall that if $\Omega \subseteq \mathbb{C}$ and X is a Banach space, then to verify that a function $F : \Omega \to X$ is analytic it suffices to verify that $z \mapsto \phi(F(z))$ is scalar-valued analytic for any $\phi \in X^*$ (see [18, p. 266, Theorem 1.1]). If $B(X)$ is the algebra of bounded linear operators on a Banach space X, then the analyticity of $F : \Omega \to B(X)$ can be established by verifying that the function $\phi(F(z)x)$ is scalar-valued analytic for any $\phi \in X^*$ and $x \in X$ (see [18, p. 267, Theorem 1.2]). Let $w(\zeta) = (1-|\zeta|)^\beta$ and $C_{0,w}(\mathbb{D})$ be the Banach space of continuous functions $f : \mathbb{D} \to \mathbb{C}$ such that fw is bounded and vanishes on $\mathbb{T}$, with the norm of f in $C_{0,w}(\mathbb{D})$ given by $\|wf\|_\infty$. For fixed $f \in C_0(\mathbb{D})$, the mapping $A : S \to C_{0,w}(\mathbb{D})$ given by

$$z \mapsto f(\cdot)(1-|\cdot|)^{-z}$$

is analytic, because $f(\zeta)(1-|\zeta|)^{\beta-z}$ is bounded in a neighbourhood of any fixed $z \in S$ (we use that $\operatorname{Re} z < \beta$ here), so we easily see that

$$S \ni z \mapsto \int_{\mathbb{D}} f(\zeta)(1-|\zeta|)^{\beta-z} d\mu(\zeta) \in \mathbb{C}$$

is scalar-valued analytic for each finite Borel measure μ on $\mathbb{D}$. Consequently $A' : S \to C_0(\mathbb{D})$ given by

$$z \mapsto \exp\left(\frac{-\epsilon}{1-|\cdot|}\right)(Rh_z)(\cdot)$$

is analytic, because it is equal to A composed with bounded linear maps. Let $M(z) : C_0(\mathbb{D}) \to C_0(\mathbb{D})$ be the bounded linear operator of multiplication by $(1-|\cdot|)^z$. Then $M : S \to B(C_0(\mathbb{D}))$ is analytic, because

$$S \ni z \to \int_{\mathbb{D}} (1-|\zeta|)^z g(\zeta) d\mu(\zeta)$$

is clearly analytic for any finite Borel measure μ on $\mathbb{D}$ and every $g \in C_0(\mathbb{D})$. We conclude that $G_\epsilon(z) = M(z)A'(z)$ is analytic in S. □

3 Boundedness and compactness

A consequence of the relation $(A_0^{-\alpha})^{**} = A^{-\alpha}$ is that any bounded linear operator T defined on $A_0^{-\alpha}$ has an extension to $A^{-\alpha}$ which coincides with the double Banach space adjoint $T^{**} : A^{-\alpha} \to A^{-\alpha}$. The operator T^{**} preserves weak-* convergence of sequences in $A^{-\alpha}$, so that if $f \in A^{-\alpha}$, then

$$T^{**}f(z) = \lim_{r\to 1^-} T^{**}f_r(z) = \lim_{r\to 1^-} Tf_r(z).$$

Applying this to the case of $T = T_g$ acting on $A_0^{-\alpha}$, we see that its double adjoint coincides with T_g acting on $A^{-\alpha}$. It follows that if T_g is bounded or compact on $A_0^{-\alpha}$, then it is also bounded or compact on $A^{-\alpha}$. The converse is a part of the following result.

Proposition 3.1. *Let $g : \mathbb{D} \to \mathbb{C}$ be an analytic function.*

(i) The operator T_g is bounded on $A^{-\alpha}$ or $A_0^{-\alpha}$ if and only if $g \in \mathcal{B}$.

(ii) The operator T_g is compact on $A^{-\alpha}$ or $A_0^{-\alpha}$ if and only $g \in \mathcal{B}_0$.

(iii) The operator norm of T_g satisfies

$$\|T_g\|_{A^{-\alpha}} \leq \frac{\|g\|_{\mathcal{B}}}{\alpha}.$$

Proof. Part *(i)* and *(ii)* in the case of $A^{-\alpha}$, as well as part *(iii)*, have already been established in [17], thus we only need to verify *(i)* and *(ii)* in the case of $A_0^{-\alpha}$. The remark preceeding the proposition implies that if T_g is bounded on $A_0^{-\alpha}$, then it is bounded on $A^{-\alpha}$, and hence $g \in \mathcal{B}$. Conversely if $g \in \mathcal{B}$, then T_g is bounded on $A^{-\alpha}$, and it is easy to verify that it maps the polynomials into $A_0^{-\alpha}$. Indeed, if $f \in A_0^{-\alpha}$ is a polynomial, then

$$\begin{aligned}(1-|z|)^\alpha\Big|\int_0^z g'(\zeta) f(\zeta)\, d\zeta\Big| \\ \lesssim (1-|z|)^\alpha \|f\|_\infty \|g\|_{\mathcal{B}} \int_0^1 \frac{|z|}{(1-t|z|)} dt \\ \leq C(1-|z|)^\alpha \log\Big(\frac{1}{1-|z|}\Big).\end{aligned}$$

The last quantity tends to zero as $|z| \to 1$, so that $T_g f \in A_0^{-\alpha}$. Since the polynomials are dense in $A_0^{-\alpha}$ and T_g is bounded on $A^{-\alpha}$, we obtain $T_g A_0^{-\alpha} \subset A_0^{-\alpha}$. This completes the proof of *(i)*. The same reasoning, with obvious modificiations, leads to a proof of *(ii)*. □

We note the following useful consequence of Proposition 3.1 and the discussion preceeding it.

Corollary 3.2. *The spectra $\sigma(T_g|A_0^{-\alpha})$ and $\sigma(T_g|A^{-\alpha})$ coincide.*

Proof. We have verified that T_g acts boundedly on $A_0^{-\alpha}$ if and only if it acts boundedly on $A^{-\alpha}$. For any bounded linear operator T on a Banach space we have that $\sigma(T) = \sigma(T^*)$. Then the claim follows from this, since the double adjoint of T_g acting on $A_0^{-\alpha}$ equals T_g acting on $A^{-\alpha}$. □

4 Projections on L_w^∞

The purpose of this section is to establish some equivalent conditions of boundedness of the standard weighted Bergman projections on the spaces L_w^∞ which were defined in the introduction. We recall that the definition of the standard weighted Bergman projection P_δ, for $\delta > -1$, is

$$P_\delta f(z) = (\delta+1) \int_{\mathbb{D}} f(\zeta) \frac{(1-|\zeta|^2)^\delta}{(1-z\bar{\zeta})^{2+\delta}} dA(\zeta),$$

where f can be any measurable function for which the above integral makes sense. We also introduce the helpful sublinear operator $\tilde{P}$:

$$\tilde{P}_\delta f(z) = (\delta+1) \int_{\mathbb{D}} |f(\zeta)| \frac{(1-|\zeta|^2)^\delta}{|1-z\bar{\zeta}|^{2+\delta}} dA(\zeta).$$

Our basic proposition on boundedness of these operators is the following.

Theorem 4.1. *Let $w : \mathbb{D} \to (0,\infty)$ be a weight and $\delta > -1$. The following are equivalent:*

(i) The operator P_δ is bounded on L^∞_w.

(ii) The operator $\tilde{P}_\delta$ is bounded on L^∞_w.

(iii) The weight w satisfies

$$\sup_{z\in\mathbb{D}} w(z) \int_{\mathbb{D}} \frac{1}{w(\zeta)} \frac{(1-|\zeta|^2)^\delta}{|1-z\bar{\zeta}|^{\delta+2}}\, dA(\zeta) < \infty.$$

Proof. $(i) \Rightarrow (ii)$: Fix any $f \in L^\infty_w$. For any $z \in \mathbb{D}$ there exists a measurable function $u_{z,f} : \mathbb{D} \to \mathbb{C}$ with $|u_{z,f}(\zeta)| = 1$ for $\zeta \in \mathbb{D}$ such that

$$\frac{|f(\zeta)|}{|1-z\bar{\zeta}|^{2+\delta}} = \frac{f(\zeta)u_{z,f}(\zeta)}{(1-z\bar{\zeta})^{2+\delta}}.$$

Let $C > 0$ be the operator norm of P_δ on L^w_∞. Then we have

$$w(z)|\tilde{P}_\delta f(z)| = w(z)|P_\delta f u_{z,f}(z)| \leq C\|wfu_{z,f}\|_\infty = C\|wf\|_\infty$$

which shows that $\tilde{P}_\delta$ is bounded on L^∞_w.
$(ii) \Rightarrow (iii)$: The function w^{-1} is in L^∞_w, and so the boundedness of $\tilde{P}_\delta$ implies that

$$\sup_{z\in\mathbb{D}} w(z)\tilde{P}_\delta w^{-1}(z) < \infty,$$

which is precisely the condition (iii).
$(iii) \Rightarrow (i)$: For any $f \in L^\infty_w$ we have

$$w(z)|P_\delta f(z)| \leq \|wf\|_\infty w(z) \int_{\mathbb{D}} \frac{1}{w(\zeta)} \frac{(1-|\zeta|^2)^\delta}{|1-z\bar{\zeta}|^{2+\delta}}\, dA(\zeta).$$

Hence if (iii) holds, then P_δ is bounded on L^∞_w. □

With the investigation of spectra of T_g and weights of the form

$$w(z) = \left|e^{\frac{g(z)}{\lambda}}\right|(1-|z|^2)^\alpha$$

in mind, we will restrict our further investigation to a class of weights which share some crucial properties with the above. Therefore, we will additionally assume that our weights are bounded, differentiable and satisfy

$$(1-|z|)|\nabla w(z)| \leq k_w w(z) \tag{5}$$

for some constant $k_w > 0$. Here ∇w denotes the gradient of the function w. An easily verifiable property of such weights is that they are approximately constant on discs of the form

$$D_z = \left\{\zeta \in \mathbb{D} : |\zeta - z| < (1-|z|)/2\right\},$$

i.e., there exists a constant $C > 0$, independent of z, such that $w(\zeta)/w(s) \leq C$ whenever $\zeta, s \in D_z$. We also have the growth estimates

$$w(0)(1-|z|)^{k_w} \lesssim w(z) \lesssim w(0)(1-|z|)^{-k_w}. \tag{6}$$

We refer to [2] for proofs of the above claims, where weights satisfying (5) have been studied in the context of T_g acting on the weighted Bergman spaces.

For w satisfying the estimate (5) we will now extend Theorem 4.1 to include a fourth equivalent condition, one which will be important in the next section.

Proposition 4.2. *Let $w : \mathbb{D} \to (0,\infty)$ be a bounded and differentiable weight which satisfies (5). If for some $\delta > -1$ the projection P_δ is a bounded operator on L^∞_w, then for each integer $n \geq 1$ we have that*

$$\sup_{z\in\mathbb{D}} w(z)|f(z)| \sim \sup_{z\in\mathbb{D}} (1-|z|^2)^n w(z)|f^{(n)}(z)| + \sum_{k=0}^{n-1} |f^{(k)}(0)|$$

for all $f \in H^\infty_w$, i.e., for each integer $n \geq 1$ the right-hand side above defines an equivalent norm on H^∞_w.

Proof. We start by establishing the proposition in the special case that $n = 1$. One of the norm inequalities holds without any assumption on the boundedness of P_δ on L^∞_w, we need only the fact that w satisfies (5). Indeed, the value $w(z)$ is comparable to values of w on the circle

$$C_z = \big\{\zeta \in \mathbb{D} : |\zeta - z| = (1-|z|)/2\big\},$$

and so from Cauchy's integral formula we have that

$$|f'(z)| \lesssim \frac{\sup_{\zeta\in C_z} |f(\zeta)|}{1-|z|} \sim \frac{\sup_{\zeta\in C_z} |f(\zeta)w(\zeta)|}{(1-|z|)w(z)},$$

which clearly implies that

$$(1-|z|)w(z)|f'(z)| \lesssim \|wf\|_\infty.$$

We proceed to establish the reverse inequality. We claim that boundedness of P_δ on L^∞_w implies that the operator P^1_δ given by

$$P^1_\delta h(z) = \int_{\mathbb{D}} h(\zeta) \frac{(1-|\zeta|^2)^\delta}{\bar{\zeta}(1-z\bar{\zeta})^{2+\delta}}\, dA(\zeta)$$

is bounded on L^∞_w. To see this, we recall that by Theorem 4.1 the operator $\tilde{P}_\delta$ is also bounded on L^∞_w, and we estimate

$$w(z)|P^1_\delta h(z)| \le w(z)\Big(\int_{|\zeta|>1/2} + \int_{|\zeta|<1/2} |h(\zeta)| \frac{(1-|\zeta|^2)^\delta}{|\zeta|(1-z\overline{\zeta})^{2+\delta}}\, dA(\zeta)\Big)$$
$$\lesssim \|hw\|_\infty + \int_{|\zeta|<1/2} \frac{|h(\zeta)|}{|\zeta|}\, dA(\zeta)$$
$$\lesssim \|hw\|_\infty + \|hw\|_\infty \int_{|\zeta|<1/2} \frac{1}{|\zeta|}\, dA(\zeta) \lesssim \|hw\|_\infty$$

where in the next-to-last step we used that w is strictly positive, and hence bounded from below for $|\zeta| < 1/2$. This shows boundedness of P^1_δ on L^∞_w. Now fix a function $f \in H^\infty_w$ and let

$$h(\zeta) = f'(\zeta)(1-|\zeta|^2) \in L^\infty_w,$$
$$g(z) = \int_{\mathbb{D}} f'(\zeta) \frac{(1-|\zeta|^2)^{\delta+1}}{\overline{\zeta}(1-z\overline{\zeta})^{2+\delta}}\, dA(\zeta) = P^1_\delta h(z).$$

Differentiating g we obtain

$$g'(z) = (\delta+2) \int_{\mathbb{D}} f'(\zeta) \frac{(1-|\zeta|^2)^{\delta+1}}{(1-z\overline{\zeta})^{3+\delta}}\, dA(\zeta) = f'(z),$$

where the last equality is the reproducing property of $P_{\delta+1}$. Hence

$$f(z) - f(0) = g(z),$$

because by direct calculation we see that $g(0) = 0$. Then the boundedness of P^1_δ gives

$$\|wP^1_\delta h\|_\infty = \|w(f - f(0))\|_\infty \le \|wh\|_\infty$$

which obviously implies

$$\sup_{z\in\mathbb{D}} w(z)|f(z)| \lesssim \sup_{z\in\mathbb{D}} (1-|z|^2)w(z)|f'(z)| + |f(0)|.$$

The proof of the case $n = 1$ is complete.

To prove the theorem for $n > 1$, note that if P_δ is bounded on L^∞_w, then $P_{\delta+1}$ is bounded on $L^\infty_{\tilde{w}}$, where $\tilde{w}(\zeta) = (1-|\zeta|^2)w(\zeta)$. This follows from Theorem 4.1, because we have

$$\tilde{w}(z) \int_{\mathbb{D}} \frac{1}{\tilde{w}(\zeta)} \frac{(1-|\zeta|^2)^{\delta+1}}{|1-z\overline{\zeta}|^{\delta+3}} dA(\zeta) = w(z)(1-|z|^2) \int_{\mathbb{D}} \frac{1}{w(\zeta)} \frac{(1-|\zeta|^2)^{\delta}}{|1-z\overline{\zeta}|^{\delta+3}} dA(\zeta)$$
$$\lesssim w(z) \int_{\mathbb{D}} \frac{1}{w(\zeta)} \frac{(1-|\zeta|^2)^{\delta}}{|1-z\overline{\zeta}|^{\delta+2}} dA(\zeta).$$

The case $n > 1$ of the proposition now follows readily by induction. □

We proceed to prove the converse of Proposition 4.2. For this, we will need the following very useful result from [2, Lemma 3.2].

Lemma 4.3. *Let $w : \mathbb{D} \to (0, \infty)$ be a differentiable weight which satisfies* (5). *If $\alpha+1 > k_w$ and $\beta > \alpha + 2 + k_w$, then*

$$\int_{\mathbb{D}} w(\zeta)\frac{(1-|\zeta|^2)^\alpha}{|1-z\overline{\zeta}|^\beta}dA(\zeta) \lesssim \frac{w(z)}{(1-|z|)^{\beta-\alpha-2}}.$$

Proposition 4.4. *Let $w : \mathbb{D} \to (0, \infty)$ be a bounded and differentiable weight which satisfies* (5). *If for each $n \geq 1$*

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^n w(z)|f^{(n)}(z)| + \sum_{k=1}^{n-1} |f^{(k)}(0)|$$

defines an equivalent norm on H^∞_w, then there exists a $\delta > -1$ such that P_δ is bounded on L^∞_w.

Proof. Since w satisfies (6), we see that for sufficiently large δ the integral defining $P_\delta f$ converges for every $f \in L^\infty_w$. We will show that δ can be chosen so that the operator P_δ is bounded on L^∞_w. This will follow from the assumption if we show that for some $n \geq 0$ we have

$$w(z)(1-|z|^2)^n \left| \int_{\mathbb{D}} f(\zeta)\frac{(1-|\zeta|^2)^\delta}{(1-z\overline{\zeta})^{\delta+2+n}}\, dA(\zeta)\right| \leq C\|fw\|_\infty.$$

We have the obvious estimate

$$\left| \int_{\mathbb{D}} f(\zeta)\frac{(1-|\zeta|^2)^\delta}{(1-z\overline{\zeta})^{\delta+2+n}}\, dA(\zeta)\right| \leq \|fw\|_\infty \int_{\mathbb{D}} \frac{1}{w(\zeta)}\frac{(1-|\zeta|^2)^\delta}{|1-z\overline{\zeta}|^{\delta+2+n}}\, dA(\zeta).$$

An easy computation shows that the weight $\tilde{w} = 1/w$ also satisfies (5), so if n and δ are sufficiently large, then by Lemma 4.3 we obtain

$$\int_{\mathbb{D}} \frac{1}{w(\zeta)}\frac{(1-|\zeta|^2)^\delta}{|1-z\overline{\zeta}|^{\delta+2+n}}\, dA(\zeta) \leq C\frac{1}{w(z)(1-|z|)^n}. \qquad \square$$

Proposition 4.2 and Proposition 4.4 now imply the following version of Theorem 4.1.

Theorem 4.5. *Let $w : \mathbb{D} \to (0, \infty)$ be a bounded and differentiable weight which satisfies* (5). *The following are equivalent:*

(i) The operator P_δ is bounded on L^∞_w for some $\delta > -1$.

(ii) The operator $\tilde{P}_\delta$ is bounded on L^∞_w for some $\delta > -1$

(iii) There exists $\delta > -1$ such that

$$\sup_{z\in\mathbb{D}} w(z) \int_{\mathbb{D}} \frac{1}{w(\zeta)} \frac{(1-|\zeta|^2)^\delta}{|1-z\overline{\zeta}|^{\delta+2}}\, dA(\zeta) < \infty.$$

(iv) For each $n \geq 1$,

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^n w(z)|f^{(n)}(z)| + \sum_{k=0}^{n-1} |f^{(k)}(0)|$$

defines an equivalent norm on H^∞_w.

5 Spectrum

As mentioned in the introduction, for fixed analytic functions $f, h : \mathbb{D} \to \mathbb{C}$, the equation

$$h - \frac{1}{\lambda} T_g h = f$$

has a unique solution $R_{\lambda,g} f$ which is given by

$$R_{\lambda,g} f(z) = f(0) e^{\frac{g(z)}{\lambda}} + e^{\frac{g(z)}{\lambda}} \int_0^z e^{-\frac{g(\zeta)}{\lambda}} f'(\zeta)\, d\zeta.$$

It is easy to see that T_g is injective if $g \neq 0$, and therefore non-zero λ is in the resolvent set of T_g acting on any Banach space X of analytic functions if and only if the operator $R_{\lambda,g}$ is bounded on X. If this is the case, then $R_{\lambda,g}$ is automatically invertible on X.

5.1 Spectrum.

We will now apply the results of Section 4 to prove Theorem 5.3, which provides a characterization of the spectrum of T_g acting on $A^{-\alpha}$.

Proposition 5.1. *The operator $R_{\lambda,g}$ is bounded on $A^{-\alpha}$ if and only if*

$$e^{g/\lambda} \in A^{-\alpha} \tag{7}$$

and the weight

$$w(z) = \left|e^{\frac{g(z)}{\lambda}}\right| (1-|z|^2)^\alpha$$

satisfies

$$\sup_{z\in\mathbb{D}} w(z)|f(z)| \sim \sup_{z\in\mathbb{D}} (1-|z|^2) w(z)|f'(z)| + |f(0)| \tag{8}$$

for every $f \in H^\infty_w(\mathbb{D})$.

Proof. Assume first that $R_{\lambda,g}$ is bounded on $A^{-\alpha}$. Then $e^{g/\lambda} = R_{\lambda,g}1 \in A^{-\alpha}$, so that (7) holds. Moreover, note that $f \in H^\infty_w$ if and only if $e^{g/\lambda} f \in A^{-\alpha}$, and that if $f(0) = 0$, then

$$e^{g/\lambda} f = R_{\lambda,g} \int e^{g/\lambda} f'.$$

Hence by the invertibility of $R_{\lambda,g}$ we have that $\int e^{g/\lambda} f' \in A^{-\alpha}$ and

$$\|e^{g/\lambda} f\|_{-\alpha} \sim \|e^{g/\lambda} f'\|_{-(\alpha+1)},$$

which is clearly equivalent to (8). Conversely, assume that (7) and (8) hold. Then $R_{\lambda,g}$ maps constants into $A^{-\alpha}$ because $R_{\lambda,g}1 = e^{g/\lambda}$, while for any $f \in A^{-\alpha}$ with $f(0) = 0$ we have

$$\begin{aligned}\|R_{\lambda,g} f\|_{-\alpha} &= \sup_{z\in\mathbb{D}} w(z) \Big| \int_0^z e^{-g(\zeta)/\lambda} f'(\zeta) d\zeta \Big| \\ &\sim \sup_{z\in\mathbb{D}} (1 - |z|^2)^{\alpha+1} |f'(z)| < \infty. \square\end{aligned}$$

We make a short remark. By Corollary 3.2, $R_{\lambda,g}$ is bounded on $A^{-\alpha}$ if and only if it is bounded on $A_0^{-\alpha}$. Since $1 \in A_0^{-\alpha}$, we see that if $R_{\lambda,g}$ is bounded on $A^{-\alpha}$, then we obtain the stronger statement that $e^{g/\lambda} \in A_0^{-\alpha}$.

Proposition 5.2. *Assume that* $0 \neq \lambda \in \rho(T_g | A^{-\alpha})$ *and let*

$$w(z) = \left|e^{\frac{g(z)}{\lambda}}\right| (1 - |z|^2)^\alpha.$$

For each $n \geq 1$,

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)^n w(z) |f^{(n)}(z)| + \sum_{k=0}^{n-1} |f^{(k)}(0)|$$

defines an equivalent norm on H^∞_w.

Proof. The assumption that $\lambda \in \rho(T_g | A^{-\alpha})$ mean precisely that $R = R_{\lambda,g}$ is bounded on $A^{-\alpha}$, and by Corollary 3.2 it is also bounded on $A_0^{-\alpha}$. For any sufficiently large $\beta > 0$ we have by Proposition 3.1 that $|\lambda| > \|T_g\|_{A^{-\beta}}$, and for such β the operator R_λ, g is bounded on $A_0^{-\beta}$. Then from Theorem 2.2 and Corollary 3.2 we obtain that $R_{\lambda,g}$ is bounded on $A^{-\beta}$ for all $\beta > \alpha$, and the claim follows from Proposition 5.1 by induction. □

We can now prove the main result characterizing the spectrum of T_g acting on $A^{-\alpha}$.

Theorem 5.3. *Assume that* $g \in \mathcal{B}$, $\lambda \in \mathbb{C} \setminus \{0\}$ *and*

$$w(z) = \left|e^{\frac{g(z)}{\lambda}}\right| (1 - |z|^2)^\alpha.$$

The following are equivalent:

(i) $\lambda \in \rho(T_g|A^{-\alpha})$.

(ii) For some $\delta > -1$, *the weight* w *satisfies*

$$\sup_{z\in\mathbb{D}} w(z) \int_{\mathbb{D}} \frac{1}{w(\zeta)} \frac{(1-|\zeta|^2)^\delta}{|1-z\overline{\zeta}|^{\delta+2}}\, dA(\zeta) < \infty.$$

Proof. The implication $(i) \Rightarrow (ii)$ is covered by Proposition 5.2 together with Theorem 4.5. If (ii) holds, then $w(z)$ is certainly bounded, and this together with Theorem 4.5 implies that the two conditions of Proposition 5.1 hold, so that (ii) implies (i). □

5.2 **Applications.** As a first application of Theorem 5.3 we establish the spectral stability property of the operator T_g that was mentioned in the introduction. Next we obtain the spectrum of T_g in the case g is the anti-derivative of a rational function. The two results are then combined in Theorem 5.6 to obtain the spectrum of T_g whenever $g = r + h + b$, where r' is rational, $h \in H^\infty$ and $b \in \mathcal{B}_0$.

Theorem 5.4. *Let* $g, h \in \mathcal{B}$ *and assume that* $\sigma(T_h|A^{-\alpha}) = \{0\}$. *Then*

$$\sigma(T_{g+h}|A^{-\alpha}) = \sigma(T_g|A^{-\alpha}).$$

Proof. It will be sufficient to show that $\rho(T_g|A^{-\alpha}) \subseteq \rho(T_{g+h}|A^{-\alpha})$, since the other inclusion follows by replacing g with $g + h$ and h with $-h$. Fix $\lambda \in \rho(T_g|A^{-\alpha})$. By Theorem 5.3 we must verify that there exists a $\delta > -1$ such that

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |e^{g(z)/\lambda}||e^{h(z)/\lambda}| \int_{\mathbb{D}} |e^{-g(\zeta)/\lambda}||e^{-h(\zeta)/\lambda}| \frac{(1-|\zeta|^2)^{\delta-\alpha}}{|1-z\overline{\zeta}|^{\delta+2}} dA(\zeta) < \infty.$$

Take $p, q > 1$ with $1/p + 1/q = 1$ and p close enough to 1 so that $\tilde{\lambda} = \lambda/p \in \rho(T_g|A^{-\alpha})$. Let $\hat{\lambda} = \lambda/q$. Use Hölder's inequality to obtain

$$\begin{aligned}
&\int_{\mathbb{D}} |e^{-g(\zeta)/\lambda}||e^{-h(\zeta)/\lambda}| \frac{(1-|\zeta|^2)^{\delta-\alpha}}{|1-z\overline{\zeta}|^{\delta+2}} dA(\zeta) \\
&\le \left(\int_{\mathbb{D}} |e^{-g(\zeta)/\tilde{\lambda}}| \frac{(1-|\zeta|^2)^{\delta-\alpha}}{|1-z\overline{\zeta}|^{\delta+2}} dA(\zeta) \right)^{1/p} \\
&\times \left(\int_{\mathbb{D}} |e^{-h(\zeta)/\hat{\lambda}}| \frac{(1-|\zeta|^2)^{\delta-\alpha}}{|1-z\overline{\zeta}|^{\delta+2}} dA(\zeta) \right)^{1/q}.
\end{aligned}$$

Now use the assumption that $\hat{\lambda} \in \rho(T_h|A^{-\alpha}) = \mathbb{C} \setminus \{0\}$ and Theorem 5.3 to see that $\delta > -1$ can be chosen big enough so that we simultaneously have

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |e^{g(z)/\tilde\lambda}|\int_{\mathbb{D}}|e^{-g(\zeta)/\tilde\lambda}|\frac{(1-|\zeta|^2)^{\delta-\alpha}}{|1-z\overline{\zeta}|^{\delta+2}}dA(\zeta)=C_1<\infty$$

and

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |e^{h(z)/\hat\lambda}|\int_{\mathbb{D}}|e^{-h(\zeta)/\hat\lambda}|\frac{(1-|\zeta|^2)^{\delta-\alpha}}{|1-z\overline{\zeta}|^{\delta+2}}dA(\zeta)=C_2<\infty.$$

We get that

$$\begin{aligned}
&(1-|z|^2)^\alpha |e^{g(z)/\lambda}||e^{h(z)/\lambda}|\int_{\mathbb{D}}|e^{-g(\zeta)/\lambda}||e^{-h(\zeta)/\lambda}|\frac{(1-|\zeta|^2)^{\delta-\alpha}}{|1-z\overline{\zeta}|^{\delta+2}}dA(\zeta)\\
&\le (1-|z|^2)^\alpha |e^{g(z)/\lambda}||e^{h(z)/\lambda}|\left(\int_{\mathbb{D}}|e^{-g(\zeta)/\tilde\lambda}|\frac{(1-|\zeta|^2)^{\delta-\alpha}}{|1-z\overline{\zeta}|^{\delta+2}}dA(\zeta)\right)^{1/p}\\
&\qquad\times\left(\int_{\mathbb{D}}|e^{-h(\zeta)/\hat\lambda}|\frac{(1-|\zeta|^2)^{\delta-\alpha}}{|1-z\overline{\zeta}|^{\delta+2}}dA(\zeta)\right)^{1/q}\\
&=\left((1-|z|^2)^\alpha |e^{g(z)/\tilde\lambda}|\int_{\mathbb{D}}|e^{-g(\zeta)/\tilde\lambda}|\frac{(1-|\zeta|^2)^{\delta-\alpha}}{|1-z\overline{\zeta}|^{\delta+2}}dA(\zeta)\right)^{1/p}\\
&\quad\times\left((1-|z|^2)^\alpha |e^{h(z)/\hat\lambda}|\int_{\mathbb{D}}|e^{-h(\zeta)/\hat\lambda}|\frac{(1-|\zeta|^2)^{\delta-\alpha}}{|1-z\overline{\zeta}|^{\delta+2}}dA(\zeta)\right)^{1/q}\\
&\le C_1^{1/p}C_2^{1/q}.\Box
\end{aligned}$$

Corollary 5.5. *Let $g, h \in \mathcal{B}$ and assume that $h \in H^\infty$ or $h \in \mathcal{B}_0$. Then*

$$\sigma(T_{g+h}|A^{-\alpha}) = \sigma(T_g|A^{-\alpha}).$$

Proof. By Proposition 3.1 if $h \in \mathcal{B}_0$, then T_h is compact and hence $\sigma(T_h|A^{-\alpha}) = \{0\}$ since T_g has no eigenvalues. On the other hand if $h \in H^\infty$ then for all $\lambda \neq 0$ the function $e^{h/\lambda}$ is bounded from above and below in $\mathbb{D}$. It then follows from Proposition 2.1 that condition (ii) of Theorem 5.3 is satisfied, so that again $\sigma(T_h|A^{-\alpha}) = \{0\}$. The claim then follows from Theorem 5.4. □

Let $\omega_0, \ldots, \omega_n$ be distinct points on the circle $\mathbb{T}$. For non-zero complex numbers $c_0, \ldots, c_n$ let

$$g(z) = \sum_{k=0}^{n} c_k \log\left(\frac{1}{1-\overline{\omega_k}z}\right) \in \mathcal{B}.$$

The spectrum of T_g in this case turns out to be

$$\sigma(T_g|A^{-\alpha}) = \bigcup_{k=1}^{n} \Big\{\lambda \in \mathbb{C} : \mathrm{Re}(c_k/\lambda) \geq \alpha\Big\}. \tag{9}$$

This result can be predicted from results in [4], where it is shown that the equality above holds with T_g replaced by the operator $f \mapsto \frac{1}{z}T_g f$. We shall therefore not carry out the entire argument, but merely indicate how condition (ii) of Theorem 5.3 can be used to establish (9). Let $s_k = \mathrm{Re}(c_k/\lambda)$. Since for each k the function $\log\left(\frac{1}{1-\overline{\omega_k}z}\right)$ has bounded imaginary part, it follows easily that

$$|e^{g(z)/\lambda}| \sim \prod_{k=0}^{n} |1-\overline{\omega_k}z|^{-s_k}. \tag{10}$$

If for some k we have that $s_k > \alpha$, then by the above $e^{g/\lambda} \notin A^{-\alpha}$, which by Proposition 5.1 implies that

$$\bigcup_{k=1}^{n} \Big\{\lambda \in \mathbb{C} : \mathrm{Re}(c_k/\lambda) \geq \alpha\Big\} \subseteq \sigma(T_g : A^{-\alpha}).$$

Conversely, if $s_k < \alpha$ for $k = 0, \ldots, n$, then to show that $\lambda \in \rho(T_g|A^{-\alpha})$ it will suffice by (10) and (ii) of Theorem 5.3 to show that

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha} \prod_{k=0}^{n} |1-\overline{\omega_k}z|^{-s_k} \int_{\mathbb{D}} \prod_{k=0}^{n} |1-\overline{\omega_k}\zeta|^{s_k} \frac{(1-|\zeta|^2)^{\delta-\alpha}}{|1-z\overline{\zeta}|^{\delta+2}} dA(\zeta) < \infty$$

holds for some $\delta > -1$. This follows from Proposition 2.1 together with a straightforward and elementary computation involving isolation of the possible poles of the functions $|1-\overline{\omega}_k\zeta|^{s_k}$.

Theorem 5.6. *Let $h \in H^{\infty}$, $b \in \mathcal{B}_0$ and*

$$r(z) = \sum_{k=0}^{n} c_k \log\Big(\frac{1}{1-\overline{\omega_k}z}\Big).$$

If $g = r + h + b$, then

$$\begin{aligned}\sigma(T_g|A^{-\alpha}) &= \bigcup_{k=1}^{n} \Big\{\lambda \in \mathbb{C} : \mathrm{Re}(c_k/\lambda) \geq \alpha\Big\} \\ &= \{0\} \cup \overline{\Big\{\lambda \in \mathbb{C}\setminus\{0\} : e^{g/\lambda} \notin A^{-\alpha}\Big\}}.\end{aligned}$$

Proof. The first of the two set equalities in the statement is immediate from the discussion preceeding the theorem and Corollary 5.5. To establish the second equality it will be sufficient to show that $e^{g/\lambda} \notin A^{-\alpha}$ whenever λ is in the interior of one of the closed disks whose union is $\sigma(T_g|A^{-\alpha})$. For any such λ there exists an $\epsilon > 0$ such that $e^{r/\lambda} \notin A^{-\alpha-\epsilon}$. Since $h \in H^\infty$ and $b \in \mathcal{B}_0$, we obtain easily that

$$e^{(h(z)+b(z))/\lambda} \gtrsim (1-|z|^2)^\epsilon.$$

Then

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |e^{g(z)/\lambda}| \gtrsim \sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha+\epsilon} |e^{r(z)/\lambda}| = \infty,$$

so $e^{g/\lambda} \notin A^{-\alpha}$. □

5.3 An important example. For $p > 1$ and $\eta > -1$ the class $B_p(\eta)$ consists of weights w on $\mathbb{D}$ for which there exists a constant $C > 0$ such that

$$\Big(\int_{S(\theta,h)} w(\zeta)(1-|\zeta|^2)^\eta dA(\zeta)\Big)\Big(\int_{S(\theta,h)} w(\zeta)^{-q/p}(1-|\zeta|^2)^\eta dA(\zeta)\Big)^{p/q} \le Ch^{p(\eta+2)}$$

for any Carleson box $S(\theta, h)$ given by

$$S(\theta,h) = \{z = re^{it} \in \mathbb{D} : 1-r < h, |t-\theta| < h\}.$$

The significance of this definition comes from the work of Békollé, who proves that the $B_p(\eta)$-condition is related to the boundedness of the standard weighted Bergman projections on weighted Bergman spaces. See [7] for details.

Proposition 5.7. *Let $w : \mathbb{D} \to (0,\infty)$ be a weight. If w satisfies for some $\delta > -1$ the condition (ii) of Theorem 5.3, then*

$$w(z)(1-|z|^2)^{-\eta} \in B_2(\eta)$$

for $\eta = \delta/2$.

Proof. Let $S = S(\theta, h)$ be a Carleson square of width $h > 0$. We have

$$w(z)\int_S \frac{1}{w(\zeta)} \frac{(1-|\zeta|^2)^\delta}{|1-z\overline{\zeta}|^{\delta+2}} dA(\zeta) \le C$$

for all $z \in \mathbb{D}$, and in particular for $z \in S$. But if $z, \zeta \in S$, then $|1 - z\overline{\zeta}| \lesssim h$, and consequently

$$w(z)\int_S \frac{1}{w(\zeta)}(1-|\zeta|^2)^\delta dA(\zeta) \lesssim h^{\delta+2}.$$

Now integrate for $z \in S$ to obtain

$$\int_S w(z)dA(z) \int_S \frac{1}{w(\zeta)}(1-|\zeta|^2)^\delta dA(\zeta) \lesssim h^{\delta+4}.$$

Setting $\eta = \delta/2$, the last inequality means precisely that $w(z)(1-|z|^2)^{-\eta} \in B_2(\eta)$. $\square$

The significance of the above proposition in the study of the spectrum $\sigma(T_g|A^{-\alpha})$ is the following. We have seen in this section that the equality

$$\sigma(T_g|A^{-\alpha}) = \{0\} \cup \overline{\left\{\lambda \in \mathbb{C} \setminus \{0\} : e^{g/\lambda} \notin A^{-\alpha}\right\}} \tag{11}$$

holds for a large class of symbols g. The fact that (11) does not hold for general $g \in \mathcal{B}$ can be seen from Proposition 5.7 and the corresponding result for weighted Bergman spaces $L_a^{p,\alpha}$. For $p > 0$ and $\alpha > -1$ the space $L_a^{p,\alpha}$ consists of functions analytic in $\mathbb{D}$ which satisfy

$$\|f\|_{p,\alpha}^p = \int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^\alpha dA(z) < \infty.$$

The operator T_g acts boundedly on $L_a^{p,\alpha}$ if and only if $g \in \mathcal{B}$, and it is shown in [2] that $\lambda \in \rho(T_g|L_a^{p,\alpha})$ if and only if the weight

$$w(z) = |e^{pg(z)/\lambda}|(1-|z|^2)^\alpha$$

is integrable (which means precisely that $e^{g/\lambda} \in L_a^{p,\alpha}$) and $\tilde{w}(z) = w(z)(1-|z|)^{-\eta}$ satisfies for some $\eta > -1$ and $p_0 > 1$ the $B_{p_0}(\eta)$-condition. It is shown in [2] that there exists a function $g \in \mathcal{B}$ such that $e^{g/\lambda}$ belongs to the space $L_a^{p,\alpha}$ for all $\lambda \in \mathbb{C} \setminus \{0\}$ and $p > 0$, yet the spectrum $\sigma(T_g|L_a^{p,\alpha})$ is always larger than $\{0\}$. See [2, Section 5] for details of the construction of the function g and its properties. It can be see in [2] that g also satisfies $e^{g/\lambda} \in A_0^{-\alpha}$ for all $\lambda \in \mathbb{C} \setminus \{0\}$ and all $\alpha > 0$. Let $w(z) = |e^{g(z)/\lambda}|(1-|z|^2)^\alpha$. If $\lambda \in \rho(T_g|A^{-\alpha})$, then condition (ii) of Theorem 5.3 holds for w for sufficiently large δ, and consequently by Proposition 5.7 and the results of [2] mentioned above we have that $\lambda \in \rho(T_g|L_a^{1,\alpha})$. It follows that

$$\sigma(T_g|A^{-\alpha}) \supseteq \sigma(T_g|L_a^{1,\alpha}).$$

Since $\sigma(T_g|L_a^{1,\alpha})$ is bigger than $\{0\}$, thus so is $\sigma(T_g|A^{-\alpha})$.

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Paper II

Generalized Cesàro operators: geometry of spectra and quasi-nilpotency

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Abstract

For the class of Hardy spaces and standard weighted Bergman spaces of the unit disk we prove that the spectrum of a generalized Cesàro operator T_g is unchanged if the symbol g is perturbed to $g + h$ by an analytic function h inducing a quasi-nilpotent operator T_h, i.e. spectrum of T_h equals $\{0\}$. We also show that any T_g operator which can be approximated in the operator norm by an operator T_h with bounded symbol h is quasi-nilpotent. In the converse direction, we establish an equivalent condition for the function $g \in$ **BMOA** to be in the **BMOA**-norm closure of H^∞. This condition turns out to be equivalent to quasi-nilpotency of the operator T_g on the Hardy spaces. This raises the question whether similar statement is true in the context of Bergman spaces and the Bloch space. Furthermore, we provide some general geometric properties of the spectrum of T_g operators.

1 Introduction

Let $\mathbb{D}$ denote the unit disk of the complex plane $\mathbb{C}$. The generalized Cesàro operator with analytic symbol $g : \mathbb{D} \to \mathbb{C}$ acts on an analytic function $f : \mathbb{D} \to \mathbb{C}$ by

$$T_g f(z) = \int_0^z f(\zeta) g'(\zeta)\, d\zeta, \quad z \in \mathbb{D}.$$

We will for the most part be working in the context of T_g acting on the Hardy spaces and the standard weighted Bergman spaces. It is a classical result in the theory that T_g is bounded on the Hardy spaces H^p, for $0 < p < \infty$, if and only if the symbol g lies in the space **BMOA** (see [6]). The Hardy spaces H^p are defined, as usual, to be the spaces of analytic functions in $\mathbb{D}$ which satisfy

$$\|f\|_{H^p}^p := \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^p \, dm(\zeta) = \int_{\mathbb{T}} |f(\zeta)|^p dm(\zeta) < \infty,$$

where dm denotes the normalized Lebesgue measure on the unit circle $\mathbb{T} = \{z : |z| = 1\}$ and the last integral is defined by the boundary values of f. The space **BMOA** is the dual of H^1 under the usual Cauchy pairing

$$\lim_{r\to 1^-} \int_{\mathbb{T}} f(\zeta)\overline{g(r\zeta)}dm(\zeta), \quad f \in H^1, g \in \mathbf{BMOA}.$$

The weighted Bergman space $L_a^{p,\alpha}$, for $0 < p < \infty$ and $\alpha > -1$, consists of analytic functions in $\mathbb{D}$ which satisfy

$$\|f\|_{L^{p,\alpha}}^p := \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z) < \infty,$$

where dA denotes the normalized area measure. In this case, the criterion for boundedness of T_g is that the symbol g belongs to the Bloch space $\mathcal{B}$ (see, for instance, [2]), which is the space of analytic functions in $\mathbb{D}$ satisfying

$$\sup_{z\in\mathbb{D}} (1-|z|^2)|g'(z)| < \infty.$$

The purpose of the present paper is to shed light on some spectral properties of T_g operators, with special emphasis on the quasi-nilpotent operators.

We start by briefly describing a common approach that has been followed with the purpose of characterizing the spectrum of T_g operators for several classes of Banach spaces X of analytic functions. We will always implicitly assume that the evaluations $f \mapsto f(\lambda)$ are bounded on any space considered. Since $T_g f(0) = 0$ for any g and f, if X contains functions which do not vanish at $z = 0$, then T_g is not surjective, and thus the point 0 is always contained in the spectrum. It is also easy to see that the operator T_g has no eigenvalues, and that for any $\lambda \in \mathbb{C}\setminus\{0\}$ and any analytic h, the equation $(I - \lambda^{-1}T_g)f = h$ has the unique solution

$$f(z) = R_g(\lambda)h(z) := h(0)e^{\frac{g(z)}{\lambda}} + e^{\frac{g(z)}{\lambda}} \int_0^z e^{-\frac{g(\zeta)}{\lambda}} h'(\zeta)\, d\zeta. \tag{1}$$

Therefore, the operator $T_g - \lambda I$ is invertible on a space X precisely when the operator $R_g(\lambda) : X \to X$ defined above is bounded on X. Of course, this implies the norm equivalence $\|f\|_X \simeq \|R_g(\lambda)f\|_X$ for $f \in X$. This simple but crucial observation has been fruitfully employed in the study of the spectrum in [2], and later in [3] and [11], where connections have been established between the boundedness of $R_g(\lambda)$ and the theory of Muckenhoupt weights and Békollé-Bonami weights. These connections will be our principal tools when establishing the main results of this paper.

In the Hardy space setting the boundedness of the operator $R_g(\lambda)$ is equivalent to a certain weight satisfying a Muckenhoupt $\mathcal{A}_\infty$-condition. A weight for us will be a positive measurable function, and the $\mathcal{A}_\infty$-class will consist of weights w on $\mathbb{T}$ for which there exists a constant $C > 0$ such that the following estimate holds for all arcs $I \subseteq \mathbb{T}$:

$$\frac{1}{m(I)} \int_I w\, dm \leq C \exp\left(\frac{1}{m(I)} \int_I \log w\, dm \right). \tag{2}$$

Here $m(I)$ denotes the normalized Lebesgue measure of the arc I. We will also need to consider the Muckenhoupt $\mathcal{A}_2$-class, i.e the weights satisfying

$$\sup_{I\subseteq\mathbb{T}}\left(\frac{1}{m(I)}\int_I w\,dm\right)\left(\frac{1}{m(I)}\int_I w^{-1}\,dm\right)<\infty. \tag{3}$$

It is straightforward to check that $w\in\mathcal{A}_2$ if and only if $w,\ w^{-1}\in\mathcal{A}_\infty$. By the Hölder inequality, we can easily prove that $\mathcal{A}_\infty$ is closed under log-convex combinations, in the sense that whenever $w_1, w_2\in\mathcal{A}_\infty$ and $0<r<1$, then $w_1^r w_2^{1-r}\in\mathcal{A}_\infty$. See [10, Chapter 9] for more details.

The following result by Aleman and Peláez characterizes the resolvent set $\rho(T_g|H^p)$ in terms of (2).

Theorem 1.1 (Theorem C of [3]). *Assume that* $\lambda\in\mathbb{C}\setminus\{0\}$, $0<p<\infty$ *and* $g\in BMOA$. *Then, the following assertions are equivalent:*

(i) $\lambda\in\rho(T_g|H^p)$,

(ii) the weight $\exp(p\,\mathrm{Re}(g(e^{it})/\lambda))$ *satisfies the* $\mathcal{A}_\infty$*-condition.*

A similar characterization has been obtained by Aleman and Constantin in the Bergman space setting, and was further refined by Aleman, Pott and Reguera in [5]. There the Békollé-Bonami weights appear instead, which are the Bergman space counterparts of the Muckenhoupt weights. The Békollé-Bonami class B_2 consists of weights on $\mathbb{D}$ satisfying an analogue of (3):

$$\sup_{I\subseteq\mathbb{T}}\left(\frac{1}{A(S_I)}\int_{S_I} w\,dA\right)\left(\frac{1}{A(S_I)}\int_{S_I} w^{-1}\,dA\right)<\infty, \tag{4}$$

where S_I denotes the usual Carleson square associated to the arc $I\subseteq\mathbb{T}$:

$$S_I=\left\{z\in\mathbb{D}:\frac{z}{|z|}\in I,\ 1-m(I)<|z|<1\right\},$$

and $A(S_I)$ is the normalized area measure of S_I. There is an analogue of the $\mathcal{A}_\infty$-class for the Békollé-Bonami weights which is often denoted by B_∞. The class consists of weights on $\mathbb{D}$ which satisfy

$$\sup_{I\subseteq\mathbb{T}}\left(\int_{S_I} w\,dA\right)^{-1}\left(\int_{S_I} M(w1_{S_I})\,dA\right)<\infty,$$

where M denotes the usual Hardy-Littlewood maximal function over Carleson squares, and $w1_{S_I}$ is the restriction of w to the square S_I. The weights appearing in our context will be of the form $w(z) = |e^{g(z)}|$ with $g \in \mathcal{B}$, and in the case of such weights the results of [5] show that the B_∞-class can be characterized in a similar way to (2), i.e for all arcs $I \subseteq \mathbb{T}$:

$$\frac{1}{A(S_I)} \int_{S_I} w \, dA \leq C \exp\left(\frac{1}{A(S_I)} \int_{S_I} \log w \, dA \right). \tag{5}$$

Again, similarly to the Muckenhoupt weights we have that the weights w, w^{-1} satisfy (5) if and only if $w \in B_2$. Moreover, the weights satisfying (5) are closed under log-convex combinations.

The condition that appears in the characterization of the spectrum of T_g operators acting on the weighted Bergman spaces is the following.

Theorem 1.2 (Part of Theorem A of [2], Corollary 4.6 of [5]). *Let $p > 0$, $\alpha > -1$ and $g \in \mathcal{B}$. For $\lambda \in \mathbb{C} \setminus \{0\}$ the following are equivalent:*

(i) $\lambda \in \rho(T_g | L_a^{p,\alpha})$,

(ii) the weight $(1 - |z|^2)^\alpha \exp(p \operatorname{Re}(g(z)/\lambda))$ *satisfies the B_∞-condition.*

We will use the form of the resolvent in (1) and the conditions of Theorem 1.1 and Theorem 1.2 in the proofs of our main results. The results are stated in Section 2, together with a discussion. The proofs are deferred to Section 3.

2 Main results

Our first main result is a spectral stability property. This is a version of a result which appears in context of so-called growth classes in [11, Theorem 5.4].

Theorem 2.1. *Let g, h be analytic functions such that $T_g, T_h : X \to X$ are bounded, where $X = H^p$ or $X = L_a^{p,\alpha}$ and $0 < p < \infty$. Suppose that the spectrum $\sigma(T_h|X)$ equals $\{0\}$. Then*

$$\sigma(T_{g+h}|X) = \sigma(T_g|X).$$

The above stability property has previously been studied and established only in very special cases, for instance when g' is a rational function, and h is bounded or induces a compact operator T_h. See, for instance, [13, Theorem 5.2], [2, Theorem B] or [1, Theorem 2.6]. All these cases are covered by Theorem 2.1.

Our next result extends the applicability of Theorem 2.1 by identifying a large class of quasi-nilpotent T_g operators. The result holds true in a much larger class of spaces than just the Hardy and Bergman spaces.

Theorem 2.2. *Let X be a Banach space of analytic functions in $\mathbb{D}$ which contains the constants and such that the algebra $B(X)$ of bounded linear operators on X contains the multiplication operators M_h and the generalized Cesàro operators T_h whenever $h \in H^\infty$. Then we have that $\sigma(T_g|X) = \{0\}$ whenever T_g lies in the norm-closure of $\{T_h : h \in H^\infty\}$ in $B(X)$.*

The conclusion of Theorem 2.2 also holds in the case of the metric spaces H^p and $L_a^{p,\alpha}$ for $p \in (0,1)$, as will be clear from the proof given in Section 3. The result is particularly useful in case that the space of symbols inducing bounded operators is known, as is the case for the Hardy and Bergman spaces. For instance, the following consequence is immediate from the well-known norm comparabilities $\|T_g\|_{H^p} \simeq \|g\|_{\mathbf{BMOA}}$ and $\|T_g\|_{L_a^{p,\alpha}} \simeq \|g\|_{\mathcal{B}}$.

Corollary 2.3. *If g lies in the norm-closure of H^∞ in BMOA or in the norm-closure of H^∞ in $\mathcal{B}$, then we have that $\sigma(T_g|H^p) = \{0\}$ and that $\sigma(T_g|L_a^{p,\alpha}) = \{0\}$, respectively.*

It is natural to ask what can be said about the converse statement. If $g \in$ **BMOA** or $g \in \mathcal{B}$, and the operator T_g is quasi-nilpotent on H^p or on $L_a^{p,\alpha}$, is then g necessarily contained in the closure of H^∞ in **BMOA**, or $\mathcal{B}$, respectively? The Bergman case is related to a long-standing open problem which will be discussed below. In the case of the Hardy spaces the converse does hold. In fact we prove a stronger statement, with a nice geometric flavour.

Theorem 2.4. *Let $0 < p < \infty$ and $g \in$ BMOA. If the spectrum $\sigma(T_g \mid H^p)$ does not contain any non-zero points of the real and imaginary axes, then g lies in the norm-closure of H^∞ in BMOA, and thus $\sigma(T_g \mid H^p) = \{0\}$.*

In the statement of Theorem 2.4 the real and imaginary axes can be replaced by two arbitrary lines which intersect orthogonally at the origin. To the authors' best knowledge, all the explicit computations of spectra of T_g on the Hardy spaces, for particular symbols g, reveal that if the spectrum is bigger than $\{0\}$, then it contains a disk with the origin on its boundary (see, for instance, [4]). This obviously implies that there are non-zero points in the spectrum which lie on either the real or imaginary axis.

Theorem 1.1 and Theorem 1.2 provide a deep link between the structure of the spectrum and the theory of exponential weights. In our last main result, we show that the log-convex property of the weight classes implies further properties of the spectrum.

Theorem 2.5. *Let $X = H^p$ or $X = L_a^{p,\alpha}$ for some $0 < p < \infty$, and $T_g : X \to X$ be bounded. For every non-zero $\lambda \in \sigma(T_g|X)$ and every $0 < r < 1$ there exists a circular arc $I_{r,\lambda}$ centered at $r\lambda$ (see Figure 1), such that the circular sector $S_{r,\lambda}$, created by taking the convex hull of the origin and $I_{r,\lambda}$ is contained in $\sigma(T_g|X)$.*

The proof of Theorem 2.4 ultimately relies on distance formulas of Jones and Garnett which are a consequence of the well-known equivalence of the $\mathcal{A}_2$-condition and the Helson-Szegö condition (see Chapter VI of [9]). A similar equivalence is not known in the case of Békollé-Bonami class B_2. In particular the missing link is the rightmost inequality

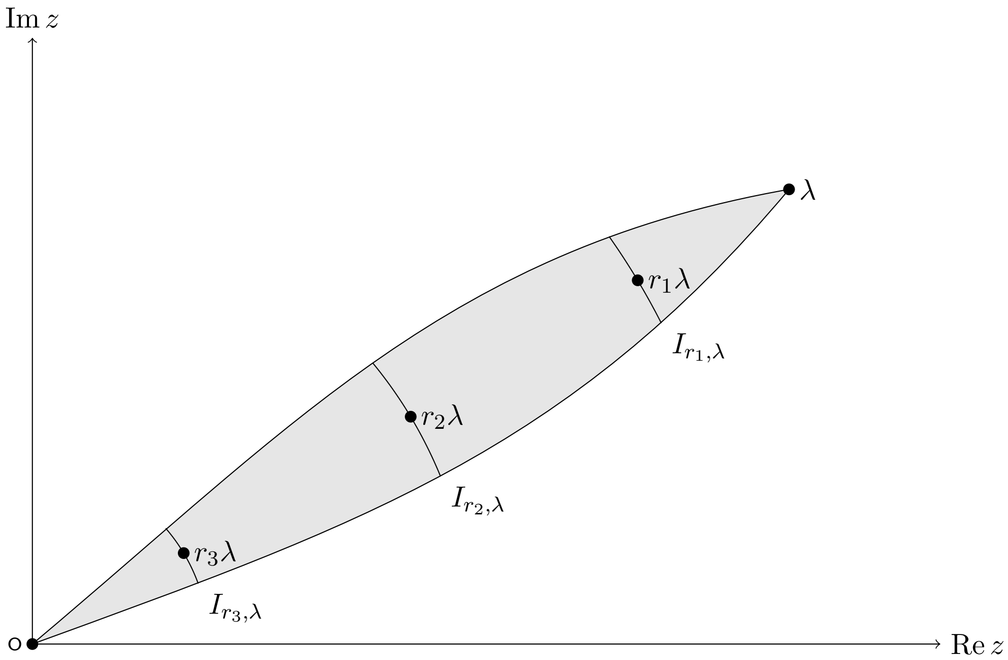

Figure 1: Points in the shaded area belong to the spectrum.

of (10) and consequently the problem of establishing the converse of Corollary 2.3 in the case of the Bergman spaces remains unsolved.

Conjecture. Let $g \in \mathcal{B}$ be such that the weights $\exp(\mathrm{Re}(g(z)/\lambda))$ satisfy (5) for all $\lambda \in \mathbb{C} \setminus \{0\}$. Then g lies in the closure of H^∞ in the Bloch norm.

With the characterization of Theorem 1.2 in mind, the conjecture asserts that the quasinilpotency of T_g on the Bergman spaces is equivalent to g lying in the closure of H^∞ in the Bloch space. The problem of characterizing this closure has been first stated in [12] and remains open to this date. However, there exist results by Galán and Nicolau [7] where a characterization of the closure of the H^p-spaces in the Bloch space is obtained in terms of square-type functions. This characterization does unfortunately not extend to H^∞, as was shown by a counterexample in [8].

3 Proofs

Proof of Theorem 2.1. We treat the case when T_g and T_h act on the H^p-spaces. We shall prove the inclusion $\rho(T_g|H^p) \subseteq \rho(T_{g+h}|H^p)$. The reverse inclusion follows by considering $g+h$ and $-h$ instead of g and h.

Fix $\lambda \in \rho(T_g|H^p)$. Then by Theorem 1.1 the weight $\exp(p\operatorname{Re}(g(e^{it})/\lambda))$ satisfies the condition (2), and since the set $\rho(T_g|H^p)$ is open, the same holds for the weight

$$w_1(e^{it}) = \exp(pp' \operatorname{Re}(g(e^{it})/\lambda))$$

whenever $p' > 1$ is sufficiently close to 1. On the other hand, the weight

$$w_2(e^{it}) = \exp(pq' \operatorname{Re}(h(e^{it})/\lambda))$$

satisfies the condition (2) for all real q' by Theorem 1.1 and the assumption that $\sigma(T_h|H^p) = \{0\}$. Let us now fix $p', q' > 1$ such that $1/p' + 1/q' = 1$ and so that w_1 satisfies (2). Let

$$w(e^{it}) = \exp(p\operatorname{Re}((g(e^{it}) + h(e^{it}))/\lambda)).$$

Note that $w = w_1^{1/p'} w_2^{1/q'}$. To show that $\lambda \in \rho(T_{g+h}|H^p)$ we now verify (2) for w:

$$\begin{aligned}
\frac{1}{m(I)}\int_I w\,dm &\le \Big(\frac{1}{m(I)}\int_I w_1\,dm\Big)^{1/p'}\Big(\frac{1}{m(I)}\int_I w_2\,dm\Big)^{1/q'} \\
\le C\exp\Big(\frac{1}{m(I)}\int_I (1/p')\log w_1\,dm\Big) &\cdot \exp\Big(\frac{1}{m(I)}\int_I (1/q')\log w_2\,dm\Big) \\
&= C\exp\Big(\frac{1}{m(I)}\int_I (1/p')\log w_1 + (1/q')\log w_2\,dm\Big) \\
&= C\exp\Big(\frac{1}{m(I)}\int_I \log w\,dm\Big).
\end{aligned}$$

This proves the theorem for the case $X = H^p$. The case $X = L_a^{p,\alpha}$ is treated in an analogous way, by using instead the characterization of Theorem 1.2, the openness of the resolvent set and Hölder's inequality. We leave out the details of the computation which are similar to the above one. □

Proof of Theorem 2.2. Fix $\lambda \in \mathbb{C}\backslash\{0\}$. Pick $h \in H^\infty$ with $\|T_g - T_h\|_{B(X)} = \|T_{g-h}\|_{B(X)}$ small enough for the operator $T_{g-h} - \lambda I$ to be invertible. For the constant function $1 \in X$ we have

$$R_g(\lambda)1 = e^{\frac{h}{\lambda}} e^{\frac{g-h}{\lambda}} = M_{e^{h/\lambda}} R_{g-h}(\lambda)1 \in X.$$

On the other hand, if $f \in X$ with $f(0) = 0$, then

$$\begin{aligned}
R_g(\lambda)f(z) &= e^{\frac{g(z)}{\lambda}} \int_0^z e^{-\frac{g(\zeta)}{\lambda}} f'(\zeta)\,d\zeta \\
&= e^{\frac{h(z)}{\lambda}} e^{\frac{g(z)-h(z)}{\lambda}} \int_0^z e^{-\frac{g(\zeta)-h(\zeta)}{\lambda}} e^{-\frac{h(\zeta)}{\lambda}} f'(\zeta)\,d\zeta \\
&= e^{\frac{h(z)}{\lambda}} e^{\frac{g(z)-h(z)}{\lambda}} \int_0^z e^{-\frac{g(\zeta)-h(\zeta)}{\lambda}} \Big[(e^{-\frac{h(\zeta)}{\lambda}} f(\zeta))' - e^{-\frac{h(\zeta)}{\lambda}} \frac{h'(\zeta)}{\lambda} f(\zeta)\Big]\,d\zeta \\
&= M_{e^{h/\lambda}} R_{g-h}(\lambda) M_{e^{-h/\lambda}} f(z) + M_{e^{h/\lambda}} R_{g-h}(\lambda) T_{e^{-h/\lambda}} f(z).
\end{aligned}$$

Note that since $h \in H^\infty$, also $e^{\pm h/\lambda} \in H^\infty$ and thus the last line above is a sum of compositions of bounded operators on X. The operator $R_g(\lambda)$ is therefore itself bounded on X. □

In order to prove Theorem 2.4, we will need a lemma which establishes the comparability of the distance to H^∞ of a **BMOA**-function in **BMO**-norm to the distance to $L^\infty(\mathbb{T})$ in **BMO**-norm. According to [9], such a result seems to originate back to the work of D. Sarason in [14]. However, we have not been able to find an explicit proof of this result. For the sake of being self-contained, we include a short proof.

Lemma 3.1. *There exists $C > 0$, such that whenever $g \in BMOA$,*

$$\inf_{f\in H^\infty} \|g - f\|_{BMO} \leq C \inf_{h\in L^\infty} \|g - h\|_{BMO} \tag{6}$$

Proof. Without loss of generality, we may assume that $g(0) = 0$. Now let $\{h_n\}_{n\in\mathbb{N}} \subset L^\infty(\mathbb{T})$ be a sequence which minimizes the **BMO**-distance from g to $L^\infty(\mathbb{T})$. By simple triangle inequality, we have for all $n \in \mathbb{N}$

$$\inf_{f\in H^\infty} \|g - f\|_{\mathbf{BMO}} \leq \|g - h_n\|_{\mathbf{BMO}} + \inf_{f\in H^\infty} \|h_n - f\|_{\mathbf{BMO}}\,. \tag{7}$$

To estimate the second norm, we use the straightforward continuous embedding $L^\infty(\mathbb{T}) \hookrightarrow$ **BMO** and the relation $\left(H^1_0\right)^\perp = H^\infty$ to get

$$\inf_{f\in H^\infty} \|h_n - f\|_{\mathbf{BMO}} \leq 2 \inf_{f\in H^\infty} \|h_n - f\|_\infty = 2 \sup_{\substack{F\in H^1_0\\ \|F\|_{H^1}\leq 1}} \left|\int_{\mathbb{T}} F h_n dm\right|. \tag{8}$$

Since the Szegö projection P is self-adjoint and $P(\overline{g}) = 0$, we can write

$$\int_{\mathbb{T}} F h_n dm = \lim_{r\to 1-} \int_{\mathbb{T}} P(F)(r\zeta) h_n(\zeta) dm(\zeta) = \lim_{r\to 1-} \int_{\mathbb{T}} F(\zeta) \overline{P(\overline{h_n - g})(r\zeta)} dm(\zeta). \tag{9}$$

Now using duality on (9) and the fact that $P : \mathbf{BMO} \to \mathbf{BMOA}$, we obtain

$$\sup_{\substack{F\in H^1_0\\ \|F\|_{H^1}\leq 1}} \left|\int_{\mathbb{T}} F h_n dm\right| \leq c \left\|P(\overline{h_n - g})\right\|_{\mathbf{BMO}} \leq c' \|h_n - g\|_{\mathbf{BMO}}\,.$$

Hence according to (8), we have established that $\inf_{f\in H^\infty} \|h_n - f\|_{\mathbf{BMO}} \lesssim \|h_n - g\|_{\mathbf{BMO}}$, thus going back to (7), we ultimately arrive at

$$\inf_{f\in H^\infty} \|g - f\|_{\mathbf{BMO}} \leq C \|g - h_n\|_{\mathbf{BMO}}\,.$$

Letting $n \to \infty$ finishes the proof of the lemma. □

Proof of Theorem 2.4. Fix $0 < p < \infty$ and suppose we have an analytic function $g \in$ **BMOA**, with the property that $\sigma(T_g|H^p) \cap (\mathbb{R} \cup i\mathbb{R}) = \{0\}$. Set $w_1 := \exp(\mathrm{Re}(g))$ and $w_2 := \exp(\mathrm{Im}(g))$. Now since $\lambda, i\lambda \in \rho(T_g|H^p)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$, Theorem 1.1 yields that the weights $w_1^{1/\lambda}, w_2^{1/\lambda}$ both satisfy the $\mathcal{A}_\infty$-condition for all $\lambda \in \mathbb{R} \setminus \{0\}$. In particular, since both the weights $w_j^{1/\lambda}, w_j^{-1/\lambda}$ satisfy the $\mathcal{A}_\infty$-condition for $j = 1, 2$, the weights $w_1^{1/\lambda}, w_2^{1/\lambda}$ are in fact $\mathcal{A}_2$-weights, for all $\lambda > 0$. A well-known consequence of the Helson-Szegö theorem (see [9, Chapter VI, Section 6]) is that there exists a constant $c > 0$ such that for all real-valued $\phi \in$ **BMO** we have

$$\frac{1}{c}\,\lambda(\phi) \leq \inf_{h \in L^\infty} \|\phi - h\|_{\mathbf{BMO}} \leq c\lambda(\phi) \tag{10}$$

where $\lambda(\phi) = \inf\left\{\lambda > 0 : e^{\phi/\lambda} \in \mathcal{A}_2\right\}$. Applying this fact to $\mathrm{Re}(g), \mathrm{Im}(g) \in$ **BMO**, we get that $\inf_{h\in L^\infty}\|\mathrm{Re}(g) - h\|_{\mathbf{BMO}} = \inf_{h\in L^\infty}\|\mathrm{Im}(g) - h\|_{\mathbf{BMO}} = 0$, thus $\inf_{h\in L^\infty}\|g - h\|_{\mathbf{BMO}} = 0$. According to Lemma 3.1, this is enough to conclude the proof.

□

Before moving on to the proof of Theorem 2.5 we establish a preliminary lemma.

Lemma 3.2. *Let* $X = H^p$ *or* $X = L^{p,\alpha}_a$ *for some* $0 < p < \infty$. *Then the spectrum* $\sigma(T_g|X)$ *is star-shaped with respect to the origin, that is, if* $\lambda \in \sigma(T_g|X)$, *then* $r\lambda \in \sigma(T_g|X)$ *for each* $r \in [0, 1]$. *In particular, the spectrum is always simply connected.*

Proof. Without loss of generality we may assume that $\sigma(T_g|X) \setminus \{0\}$ is non-trivial. Now suppose that there exists $0 < r < 1$ and a non-zero point $\lambda_0 \in \sigma(T_g|X)$, such that $r\lambda_0 \in \rho(T_g|X)$. We will show that this implies that λ_0 is not in the spectrum, and thus the claim of the lemma will be established. We shall carry out the proof in the case $X = L^{p,\alpha}_a$. According to Theorem 1.2 it suffices to establish that the weight $w_\alpha(z) = v_\alpha(z)\exp\left(p\,\mathrm{Re}(g(z)/\lambda_0)\right)$ satisfies the B_∞-condition, where $v_\alpha(z) = (1 - |z|^2)^\alpha$. We assumed that $v_\alpha \exp\left(p\,\mathrm{Re}(g/r\lambda_0))\right)$ satisfies the B_∞-condition, hence applying Hölder's inequality followed by the B_∞-condition, we obtain

$$\begin{aligned}
&\frac{1}{A(S_I)}\int_{S_I} w_\alpha dA \\
&\leq \left(\frac{1}{A(S_I)}\int_{S_I} v_\alpha \exp\left(p\,\mathrm{Re}\left(\frac{g}{r\lambda_0}\right)\right)dA\right)^r \left(\frac{1}{A(S_I)}\int_{S_I} v_\alpha dA\right)^{1-r} \\
&\lesssim \exp\left(\frac{1}{A(S_I)}\int_{S_I} \log\left[v_\alpha \exp\left(p\,\mathrm{Re}\left(\frac{g}{r\lambda_0}\right)\right)\right]dA\right)^r \left(\frac{1}{A(S_I)}\int_{S_I} v_\alpha dA\right)^{1-r}
\end{aligned}$$

$$\lesssim \exp\left(\frac{1}{A(S_I)}\int_{S_I}\log\left[v_\alpha \exp\left(p\,\mathrm{Re}\left(\frac{g}{r\lambda_0}\right)\right)\right]dA\right)^r$$

$$\times \exp\left(\frac{1}{A(S_I)}\int_{S_I}\log(v_\alpha)dA\right)^{1-r} = \exp\left(\frac{1}{A(S_I)}\int_{S_I}\log(w_\alpha)dA\right).$$

In the second last step we also used the B_∞-condition on the standard weights v_α, which follows from the simple fact that the Bergman projection P_α is bounded on $L^{p,\alpha}$. Hence by Theorem 1.2, we conclude that $\lambda_0 \in \rho(T_g|L^{p,\alpha}_a)$, which contradicts our initial assumption, thus the theorem is proved in the case $X = L^{p,\alpha}_a$. The case $X = H^p$ is more straightforward and treated analogously, using instead the characterization of Theorem 1.1. □

Proof of Theorem 2.5. This time, we carry out the proof in the case $X = H^p$, where the Bergman case is similar. Pick $\lambda \in \sigma(T_g|H^p) \setminus \{0\}$. By means of multiplying g with a unimodular constant, which corresponds to rotating the spectrum, we may without loss of generality assume that $\lambda > 0$. Now fix $0 < r < r' < 1$ and consider the circular arc parametrization $\gamma_r(t) = r\lambda e^{it}$, $-\pi < t \leq \pi$. Notice that

$$\mathrm{Re}\left(\frac{g}{\gamma_r(t)}\right) = \frac{\mathrm{Re}(g)}{r\lambda}\cos(t) + \frac{\mathrm{Im}(g)}{r\lambda}\sin(t) = \frac{\mathrm{Re}(g)}{\frac{r}{\cos(t)}\cdot\lambda} + \frac{\mathrm{Im}(g)}{\frac{r}{\sin(t)}\cdot\lambda} \tag{11}$$

Let $\left|\sigma(T_g|H^p)\right|$ denote the spectral radius of the operator T_g on H^p. Denote by $J_{r,\lambda}$ the interval consisting of t for which $\frac{r}{\cos(t)} \leq r'$ and $\frac{r(1-r')\lambda}{r'|\sin(t)|} > \left|\sigma(T_g|H^p)\right|$, i.e.

$$|t| \leq \min\left\{\arccos(r/r'), \arcsin\left(r(1-r')\lambda/r'\left|\sigma(T_g|H^p)\right|\right)\right\}.$$

Then by Theorem 1.1 and the star-shaped property of the spectrum, we have that the weight

$$u_t^{r'} := \exp\left(\frac{\mathrm{Re}(g)}{\frac{r}{r'\cos(t)}\cdot\lambda}\right) \notin \mathcal{A}_\infty \qquad \forall t \in J_{r,\lambda}, \tag{12}$$

while $v_t^{r'/(1-r')} := \exp\left(\frac{\mathrm{Im}(g)}{\frac{r(1-r')\lambda}{r'\sin(t)}}\right) \in \mathcal{A}_\infty$, for all $t \in J_{r,\lambda}$. Since the interval $J_{r,\lambda}$ is symmetric around zero and the sine-function is odd, we have that $v_{-t} = v_t^{-1}$, thus $v_t^{-r'/(1-r')}$ is also an $\mathcal{A}_\infty$-weight. Setting $w_t = \exp\left(\mathrm{Re}\left(\frac{g}{\gamma_r(t)}\right)\right)$ we can rewrite $v_t^{-1}w_t = u_t$. For the sake of obtaining a contradiction, suppose $w_{t_0} \in \mathcal{A}_\infty$, for some $t_0 \in J_{r,\lambda}$, which by Theorem 1.1 corresponds to the assumption that $\gamma_r(t_0) \in \rho(T_g|H^p)$. Now since

$v_{t_0}^{-r'/(1-r')}$ is an $\mathcal{A}_\infty$-weight and the $\mathcal{A}_\infty$-class is closed under log-convex combinations, we get that

$$u_{t_0}^{r'} = v_{t_0}^{-r'} w_{t_0}^{r'} = \left(v_{t_0}^{-r'/(1-r')}\right)^{1-r'} w_{t_0}^{r'} \in \mathcal{A}_\infty. \tag{13}$$

This contradicts our initial assumption in (12). We have thus established that the circular arc corresponding to $J_{r,\lambda}$, defined by $I_{r,\lambda} := \{r\lambda e^{it} : t \in J_{r,\lambda}\}$, is contained in $\sigma(T_g|H^p)$. Taking convex hulls of the origin and $I_{r,\lambda}$, we conclude by Lemma 3.2 that the corresponding circular sector $S_{r,\lambda}$ is included in the spectrum. □

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Paper III

Density of disc algebra functions in de Branges-Rovnyak spaces

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Abstract

We prove that functions analytic in the unit disk and continuous up to the boundary are dense in the de Branges-Rovnyak spaces induced by the extreme points of the unit ball of H^∞. Together with previous theorems it follows that this class of functions is dense in any de Branges-Rovnyak space.

1 Introduction

Let H^∞ be the algebra of bounded analytic functions in the unit disk $\mathbb{D}$ in the complex plane, and denote by $\mathcal{A}$ the disc algebra, i.e. the subalgebra of H^∞ consisting of functions which extend continuously to the closed disk. The Hardy space H^2 consists of power series in $\mathbb{D}$ with square-summable coefficients. If $\mathbb{T}$ denotes the unit circle, we identify as usual H^2 with the closed subspace of $L^2(\mathbb{T})$ consisting of functions whose negative Fourier coefficients vanish. The orthogonal projection from $L^2(\mathbb{T})$ onto H^2 is denoted by P_+.

For $\phi \in L^\infty(\mathbb{T})$ let T_ϕ denote the Toeplitz operator on H^2 defined by $T_\phi f = P_+ \phi f$. Given $b \in H^\infty$ with $\|b\|_\infty \leq 1$ we define the corresponding *de Branges-Rovnyak space* $\mathcal{H}(b)$ as

$$\mathcal{H}(b) = (1 - T_b T_{\overline{b}})^{1/2} H^2.$$

$\mathcal{H}(b)$ is endowed with the unique norm which makes the operator $(1 - T_b T_{\overline{b}})^{1/2}$ a partial isometry from H^2 onto $\mathcal{H}(b)$. Alternatively, $\mathcal{H}(b)$ is defined as the reproducing kernel Hilbert space with kernel

$$k_b(z, \lambda) = \frac{1 - \overline{b(\lambda)} b(z)}{1 - \overline{\lambda} z}.$$

$\mathcal{H}(b)$-spaces are naturally split into two classes with fairly different structures according to whether the quantity $\int_{\mathbb{T}} \log(1 - |b|)\, dm$ is finite or not. Here m denotes the normalized arc-length measure on $\mathbb{T}$. The present note concerns the approximation of $\mathcal{H}(b)$-functions by functions in $\mathcal{A} \cap \mathcal{H}(b)$ and from the technical point of view there is a major difference between the two classes, which we shall briefly explain.

If $|\int_{\mathbb{T}} \log(1 - |b|)\, dm| < \infty$, or equivalently, if b is a non-extreme point of the unit ball of H^∞, then $\mathcal{H}(b)$ contains all functions analytic in a neighborhood of the closed unit disk (see section (IV-6) of [9]). By a theorem of Sarason, the polynomials form a norm-dense subset of the space (see section (IV-3) of [9]). An interesting feature of the proofs of density of polynomials in an $\mathcal{H}(b)$-space is that the usual approach of approximating a function f first by its dilations $f_r(z) = f(rz)$, and then by their truncated Taylor series, or by their Cesàro means, does not work. Sarason's intial proof of density of polynomials is based on a duality argument. In recent years a more involved constructive polynomial approximation scheme has been obtained in [7].

The picture changes dramatically in the case when $|\int_{\mathbb{T}} \log(1 - |b|)\, dm| = \infty$, or equivalently when b is an extreme point of the unit ball of H^∞. Then it is in general a difficult task to identify any functions in the space other than the reproducing kernels, and it might happen that $\mathcal{H}(b)$ contains no non-zero function analytic in a neighborhood of the closed disk. A special class of extreme points are the inner functions. If b is inner then $\mathcal{H}(b) = H^2 \ominus bH^2$ with equality of norms, and it is a consequence of a celebrated theorem of Aleksandrov [1] that in this case the intersection $\mathcal{A} \cap \mathcal{H}(b)$ is dense in the space. The result is surprising since, as pointed out above, in most cases it is not obvious at all that $\mathcal{H}(b)$ contains any non-zero function in the disk algebra $\mathcal{A}$.

Motivated by the situation described here, E. Fricain [5], raised the natural question whether Aleksandrov's result extends to all other $\mathcal{H}(b)$-spaces induced by extreme points b of the unit ball of H^∞. It is the purpose of this note to provide an affirmative answer to this question, contained in the main result below.

Theorem 1. *If b is an extreme point of the unit ball of H^∞, then $\mathcal{A} \cap \mathcal{H}(b)$ is a dense subset of $\mathcal{H}(b)$.*

Together with Sarason's result [9] on the density of polynomials in the non-extreme case, it follows that the intersection $\mathcal{A} \cap \mathcal{H}(b)$ is dense in the space $\mathcal{H}(b)$ for any b in the unit ball of H^∞. Our proof of Theorem 1 is deferred to Section 3 and relies on a duality argument. Therefore, just as the earlier proofs of Sarason and Aleksandrov, our approach is non-constructive. Section 2 serves to establish some preliminary results.

2 Preliminaries

2.1 The norm on $\mathcal{H}(b)$. An essential step is the following useful representation of the norm in $\mathcal{H}(b)$. The authors have originally deduced the result using the techniques in [3] (see also [2, Chapter 3]), but once the goal is identified, several available techniques provide simpler proofs. For example, the proposition below can be deduced from results in [9]. For the sake of completeness, we include a new shorter proof.

Proposition 2. *Let b be an extreme point of the unit ball of H^∞ and let*

$$E = \{\zeta \in \mathbb{T} : |b(\zeta)| < 1\}.$$

Then for $f \in \mathcal{H}(b)$ the equation

$$P_+\bar{b}f = -P_+\sqrt{1-|b|^2}g.$$

has a unique solution $g \in L^2(E)$, and the map $J : \mathcal{H}(b) \to H^2 \oplus L^2(E)$ defined by

$$Jf = (f, g),$$

is an isometry. Moreover,

$$J(\mathcal{H}(b))^\perp = \Big\{(bh, \sqrt{1-|b|^2}h) : h \in H^2\Big\}.$$

Proof. Let

$$K = \Big\{(bh, \sqrt{1-|b|^2}h) : h \in H^2\Big\} \subset H^2 \oplus L^2(E)$$

and let P_1 be the projection from $H^2 \oplus L^2(E)$ onto the first coordinate H^2, i.e., $P_1(f, g) = f$. We observe first that $P_1|K^\perp$ is injective. Indeed, if $K^\perp$ contains a tuple of the form $(0, g)$, then it follows that

$$\int_{\mathbb{T}} \bar{\zeta}^n g(\zeta)\sqrt{1-|b(\zeta)|^2}\, dm(\zeta) = 0,\ n \geq 0,$$

and consequently the function $g\sqrt{1-|b|^2}$ coincides a.e. with the boundary values of the complex conjugate of a function $f \in H^2_0$. But the assumption that b is an extreme point then implies that $\int_{\mathbb{T}} \log|f|\, dm = -\infty$, and since $f \in H^2$, we conclude that $f = 0$, i.e, $g = 0$. Thus, the space $\mathcal{H} = P_1K^\perp$ with the norm $\|f\|_{\mathcal{H}} = \|P_1^{-1}f\|_{H^2\oplus L^2(E)}$ is a Hilbert space of analytic functions on $\mathbb{D}$, contractively contained in H^2, in particular, it is a reproducing kernel Hilbert space. We now show that $\mathcal{H}$ equals $\mathcal{H}(b)$ by verifying that the reproducing kernels of the two spaces coincide. This follows from a simple computation. For $\lambda \in \mathbb{D}$, the tuple

$$\begin{aligned}(f_\lambda, g_\lambda) &= \Big(\frac{1-\overline{b(\lambda)}b(z)}{1-\bar{\lambda}z}, -\frac{\overline{b(\lambda)}\sqrt{1-|b(z)|^2}}{1-\bar{\lambda}z}\Big) \\ &= \Big(\frac{1}{1-\bar{\lambda}z}, 0\Big) - \Big(\frac{\overline{b(\lambda)}b(z)}{1-\bar{\lambda}z}, \frac{\overline{b(\lambda)}\sqrt{1-|b(z)|^2}}{1-\bar{\lambda}z}\Big)\end{aligned}$$

is obviously orthogonal to K, while the last tuple on the right hand side is in K, so that f_λ is the reproducing kernel in $\mathcal{H}$, which obviously equals the reproducing kernel in $\mathcal{H}(b)$. The first assertion in the statement is now self-explanatory. □

2.2 **Cauchy transforms and two classical theorems.** The dual $\mathcal{A}'$ of the disk algebra $\mathcal{A}$ can be identified with the space $\mathcal{C}$ of Cauchy transforms of finite Borel measures on $\mathbb{T}$. The Cauchy transform $C\mu$ of a measure μ is given by

$$C\mu(z) = \int_{\mathbb{T}} \frac{1}{1 - z\overline{\zeta}} d\mu(\zeta),$$

and the duality between $\mathcal{A}$ and $\mathcal{C}$ is given by the pairing

$$\langle f, C\mu \rangle = \lim_{r \to 1^-} \int_{\mathbb{T}} f(\zeta) \overline{C\mu(r\zeta)} dm(\zeta) = \int_{\mathbb{T}} f d\overline{\mu}.$$

A proof of this fact can be found, for example, in Section 4.2 of [6]. The space $\mathcal{C}$ is endowed with the obvious quotient norm and is continuously contained in all H^p spaces for $0 < p < 1$.

Recall that analytic functions f in $\mathbb{D}$ satisfy $\sup_{0<r<1} \int_{\mathbb{T}} \log^+ |f_r| dm < \infty$ if and only if they are quotients of H^∞-functions, in particular they have finite nontangential limits a.e. on $\mathbb{T}$ which define a boundary function denoted also by f. The class $N^+(\mathbb{D})$ consists of quotients of H^∞-functions such that the denominator can be chosen to be outer. It contains in particular all Hardy spaces H^p, $p > 0$.

Two classical theorems will play an important role in the proof of Theorem 1. The first is the following theorem of Vinogradov, which also plays a crucial role in the proof of Aleksandrov's result. A proof of the below theorem the can be found in [10].

Theorem 3. *Let $f \in \mathcal{C}$. If I is an inner function such that $f/I \in N^+(\mathbb{D})$, then $f/I \in \mathcal{C}$ and $\|f/I\| \leq \|f\|$.*

The second is the Khintchin-Ostrowski theorem, and it reads as follows. A proof can be found in Section 3.2 of [8].

Theorem 4. *Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions analytic in the unit disk satisfying the following conditions:*

(i) There exists a constant $C > 0$ such that for all $n \geq 1$ we have

$$\sup_{0<r<1} \int_{\mathbb{T}} \log^+(|f_n(r\zeta)|) dm(\zeta) \leq C.$$

(ii) On some set $E \subseteq \mathbb{T}$ of positive Lebesgue measure, the sequence f_n converges in measure to a function ϕ.

Then the sequence f_n converges uniformly on compact subsets of the unit disk to a function $f \in N^+(\mathbb{D})$, and moreover $f = \phi$ a.e. on E.

3 Proof of the main result

Due to Proposition 2 we can now implement Aleksandrov's strategy from [1] which will then be combined with Theorem 3 and Theorem 4. The following result extends Alexandrov's approach to the context of $\mathcal{H}(b)$-spaces, when b is extremal in the unit ball of H^∞.

Lemma 5. *Let $b : \mathbb{D} \to \mathbb{D}$ be analytic, $E = \{\zeta \in \mathbb{T} : |b(\zeta)| < 1\}$, $B = \mathcal{A} \oplus L^2(E)$ and $B' = \mathcal{C} \oplus L^2(E)$. Then the set*

$$S = \{(C\mu, h) : C\mu/b \in N^+(\mathbb{D}), C\mu/b = h/\sqrt{1-|b|^2} \text{ a.e. on } E\}$$

is weak- closed in B'.*

Proof. Since $\mathcal{A} \oplus L^2(E)$ is separable and S is a linear subspace, it will be sufficient to show that S is weak-* sequentially closed (see Theorem 5, p.76 of [4]). Let $(C\mu_n, h_n)$ converge weak-* to $(C\mu, h)$, where $(C\mu_n, h_n) \in S$ for $n \geq 1$. Equivalently, $h_n \to h$ weakly in $L^2(E)$, and

$$\sup_n \|C\mu_n\| < \infty. \quad \lim_{n\to\infty} C\mu_n(z) = C\mu(z), \; z \in \mathbb{D}.$$

Now by passing to a subsequence and the Cesàro means of that subsequence we can assume that $h_n \to h$ in the L^2-norm. Finally, using another subsequence we may also assume that $h_n \to h$ pointwise a.e. on E. Let I_b be the inner factor of b. Since $C\mu_n/I_b \in N^+(\mathbb{D})$, it follows by Theorem 3 that $\{C\mu_n/I_b\}_{n=1}^\infty$ is a bounded sequence in $\mathcal{C}$ converging pointwise on $\mathbb{D}$ to $C\mu/I_b$. This implies weak-* convergence in $\mathcal{C}$, in particular, $C\mu/I_b \in \mathcal{C} \subset N^+(\mathbb{D})$, and consequently, $C\mu/b \in N^+(\mathbb{D})$. Moreover, we have a.e. on E that $C\mu_n/b = h_n/\sqrt{1-|b|^2}$ which converges pointwise to $h/\sqrt{1-|b|^2}$, hence we conclude that the sequence $C\mu_n$ converges in measure to some function ϕ on E. Fix any $p \in (0,1)$. Then

$$\int_{\mathbb{T}} \log^+(|C\mu_n(r\zeta)|) dm(\zeta) \lesssim \int_{\mathbb{T}} |C\mu_n(r\zeta)|^p dm(\zeta) \lesssim \sup_n \|C\mu_n\|^p < \infty.$$

Thus the assumptions of Theorem 4 are satisfied, and so (a subsequence of) $C\mu_n$ converges a.e. on E to $C\mu$. This clearly implies $C\mu/b = h/\sqrt{1-|b|^2}$ a.e. on E, i.e. $(C\mu, h) \in S$. □

We are now ready to complete the proof of the main theorem.

Proof of Theorem 1. Let J denote the embedding in Proposition 2. Based on the pairing described at the beginning of Section 2.2, a direct application of Proposition 2 gives

$$J(\mathcal{A} \cap \mathcal{H}(b)) = \cap_{h \in H^2} \ker l_h,$$

where the functionals l_h are identified with elements of $\mathcal{C} \oplus L^2(E)$ as

$$l_h = \left(hb, h\sqrt{1-|b|^2}\right).$$

It is a consequence of the Hahn-Banach theorem that the annihilator $J(\mathcal{A}\cap\mathcal{H}(b))^\perp$ is the weak-* closure of the set of the functionals l_h. Since for all $h \in H^2$ we have $l_h \in S$, the set considered in Lemma 5, by the lemma we conclude that $J(\mathcal{A}\cap\mathcal{H}(b))^\perp \subset S$. Thus if $f \in \mathcal{H}(b)$ is orthogonal to $\mathcal{A}\cap\mathcal{H}(b)$, we must have $Jf \in S$, that is

$$Jf = (hb, h\sqrt{1-|b|^2})$$

for some $h \in N^+(\mathbb{D})$. The boundary values of h satisfy

$$\int_{\mathbb{T}} |h|^2 dm = \int_{\mathbb{T}} |bh|^2 dm + \int_{\mathbb{T}} (1-|b|^2)|h|^2 dm = \|f\|^2$$

and hence by the Smirnov maximum principle we have $h \in H^2$. But then by Proposition 2, $Jf \in J(\mathcal{H}(b))^\perp$, which gives $Jf = 0$ and the proof is complete. □

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Paper IV

Hilbert spaces of analytic functions with a contractive backward shift

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Abstract

We consider Hilbert spaces of analytic functions in the disk with a normalized reproducing kernel and such that the backward shift $f(z) \mapsto \frac{f(z)-f(0)}{z}$ is a contraction on the space. We present a model for this operator and use it to prove the surprising result that functions which extend continuously to the closure of the disk are dense in the space. This has several applications, for example we can answer a question regarding reverse Carleson embeddings for these spaces. We also identify a large class of spaces which are similar to the de Branges-Rovnyak spaces and prove some results which are new even in the classical case.

1 Introduction

Let $\mathbb{D}$ be the unit disk of the complex plane $\mathbb{C}$. The present paper is concerned with the study of a class of Hilbert spaces of analytic functions on which the backward shift operator acts as a contraction. More precisely, let $\mathcal{H}$ be a Hilbert space of analytic functions such that

(A.1) the evaluation $f \mapsto f(\lambda)$ is a bounded linear functional on $\mathcal{H}$ for each $\lambda \in \mathbb{D}$,

(A.2) $\mathcal{H}$ is invariant under the backward shift operator L given by

$$Lf(z) = \frac{f(z) - f(0)}{z},$$

and we have that

$$\|Lf\|_{\mathcal{H}} \leq \|f\|_{\mathcal{H}}, \quad f \in \mathcal{H},$$

(A.3) the constant function 1 is contained in $\mathcal{H}$ and has the reproducing property

$$\langle f, 1 \rangle_{\mathcal{H}} = f(0) \quad f \in \mathcal{H}.$$

Here $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ denotes the inner product in $\mathcal{H}$. By (A.1), the space $\mathcal{H}$ comes equipped with a reproducing kernel $k_{\mathcal{H}} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ which satisfies

$$\langle f, k_{\mathcal{H}}(\cdot, \lambda) \rangle_{\mathcal{H}} = f(\lambda), f \in \mathcal{H}.$$

The condition (A.3) is a normalization condition which ensures that $k_{\mathcal{H}}(z,0) = 1$ for all $z \in \mathbb{D}$. The first example that comes to mind is a weighted version of the classical Hardy space, where the norm of an element $f(z) = \sum_{k=0}^{\infty} f_k z^n$ is given by

$$\|f\|_w^2 = \sum_{k=0}^{\infty} w_k |f_k|^2$$

with $w = (w_k)_{k=0}^n$ a nondecreasing sequence of positive numbers w_k, and $w_0 = 1$. More generally, the conditions are fulfilled in any L-invariat Hilbert space of analytic functions in $\mathbb{D}$ with a normalized reproducing kernel on which the forward shift operator M_z, given by $M_z f(z) = zf(z)$, is expansive ($\|M_z f\|_{\mathcal{H}} \geq \|f\|_{\mathcal{H}}$). Moreover, any L-invariant subspace of such a space of analytic functions satisfies the axioms as well if it contains the constants. A more detailed list of examples will be given in the next section. It includes de Branges-Rovnyak spaces, spaces of Dirichlet type and their L-invariant subspaces. Despite the conditions being rather general, it turns out that they imply useful common structural properties of these function spaces. The purpose of this paper is to reveal some of those properties and discuss their applications.

The starting point of our investigation is the following formula for the reproducing kernel. The space $\mathcal{H}$ satisfies (A.1)-(A.3) if and only if the reproducing kernel $k_{\mathcal{H}}$ of $\mathcal{H}$ has the form

$$k_{\mathcal{H}}(z,\lambda) = \frac{1 - \sum_{i\geq 1} \overline{b_i(\lambda)} b_i(z)}{1 - \overline{\lambda} z} = \frac{1 - \mathbf{B}(z)\mathbf{B}(\lambda)^*}{1 - \overline{\lambda} z}, \quad \mathbf{B}(0) = 0, \tag{1}$$

where $\mathbf{B}$ is the analytic row contraction into l^2 with entries $(b_i)_{i\geq 1}$. This follows easily from the positivity of the operator $I_{\mathcal{H}} - LL^*$ and will be proved in Proposition 2.1 below.

The representation in (1) continues to hold in the case when $\mathcal{H}$ consists of vector-valued functions, with $\mathbf{B}$ an analytic operator-valued contraction. Such kernels have been considered in [9] where they are called de Branges-Rovnyak kernels. This representation of the reproducing kernel in terms of $\mathbf{B}$ is obviously not unique. We shall denote throughout by $[\mathbf{B}]$ the class of $\mathbf{B}$ with respect to the equivalence $\mathbf{B}_1 \sim \mathbf{B}_2 \Leftrightarrow \mathbf{B}_1(z)\mathbf{B}_1^*(\lambda) = \mathbf{B}_2(z)\mathbf{B}_2^*(\lambda),\ z,\lambda \in \mathbb{D}$, and by $\mathcal{H}[\mathbf{B}]$ the Hilbert space of analytic functions satisfying (A.1)-(A.3) for which the reproducing kernel is given by (1).

Definition 1.1. The space $\mathcal{H}[\mathbf{B}]$ is of *finite rank* if there exists $\mathbf{C} \in [\mathbf{B}],\ \mathbf{C} = (c_1, c_2, \ldots)$ and an integer $N \geq 1$ with $c_n = 0,\ n \geq N$. The *rank* of $\mathcal{H}[\mathbf{B}]$ is defined as the minimal number of nonzero terms which can occur in these representations.

Note that if $\mathcal{H}[\mathbf{B}]$ is of finite rank and $\mathbf{C} \in [\mathbf{B}],\ \mathbf{C} = (c_1, \ldots, c_n, \ldots)$ has the minimum number of nonzero terms, then these must be linearly independent. The rank-zero case corresponds to the Hardy space H^2, while rank-one spaces are the classical de Branges-Rovnyak spaces. These will be denoted by $\mathcal{H}(b)$ (see [17], [18] for an accessible treatment of the theory of these spaces).

Our basic tool for the study of these spaces is a model for the contraction L on $\mathcal{H}$. The intuition comes from a simple example, namely a de Branges-Rovnyak space $\mathcal{H}(b)$ where b is a non-extreme point of the unit ball of H^∞. Then there exists an analytic outer function a such that $|b|^2 + |a|^2 = 1$ on $\mathbb{T}$, and we can consider the M_z-invariant subspace $U = \{(bh, ah) : h \in H^2\} \subset H^2 \oplus H^2$. Sarason proved (see Section IV-7 of [30]) that there is an isometric one-to-one correspondence $f \mapsto (f, g)$ between the elements of $\mathcal{H}(b)$ and the tuples in the orthogonal complement of U, and the intertwining relation $Lf \mapsto (Lf, Lg)$ holds.

One of the main ideas behind the results of this paper is that, based on the structure of the reproducing kernel, a similar construction can be carried out for any Hilbert space satisfying (A.1)-(A.3). As is to expect, in this generality the objects appearing in our model are more involved, in particular the direct sum $H^2 \oplus H^2$ needs to be replaced by the direct sum of H^2 with an M_z-invariant subspace of a vector-valued L^2-space. Details of this construction, which extend to the vector-valued case as well, will be given in Section 2. This model is essentially a special case of the functional model of Sz.-Nagy-Foias for a general contractive linear operator (see Chapter VI of [31]). Moreover, it has a connection to a known norm formula (see e.g. [5], [7], [8]) which played a key role in the investigation of invariant subspaces in various spaces of analytic functions. We explain this connection in Section 2.4.

The advantage provided by this point of view is a fairly tractable formula for the norm on such spaces. Our main result in this generality is the following surprising approximation theorem. Recall that the disk algebra $\mathcal{A}$ is the algebra of analytic functions in $\mathbb{D}$ which admit continuous extensions to clos($\mathbb{D}$).

Theorem 1.1. *Let $\mathcal{H}$ be a Hilbert space of analytic functions which satisfies (A.1)-(A.3). Then any backward shift invariant subspace of $\mathcal{H}$ contains a dense set of functions in $\mathcal{A}$.*

The proof of Theorem 1.1 is carried out in Section 3 and the argument covers the finite dimensional vector-valued case as well. Our approach is based on ideas of Aleksandrov [2] and the authors [6], however due to the generality considered here the proof is different since it avoids the use of classical theorems of Vinogradov [32] and Khintchin-Ostrowski [20, Section 3.2].

Several applications of Theorem 1.1 are presented in Section 4. One of them concerns the case when $\mathcal{H}$ satisfies (A.1)-(A.3) and, in addition, it is invariant for the forward shift. We obtain a very general Beurling-type theorem for M_z-invariant subspaces which requires the use of Theorem 1.1 in the vector-valued case.

Corollary 1.2. *Let $\mathcal{H}$ be a Hilbert space of analytic functions which satisfies (A.1)-(A.3) and is invariant for the forward shift M_z. For a closed M_z-invariant subspace $\mathcal{M}$ of $\mathcal{H}$ with $\dim \mathcal{M} \ominus M_z\mathcal{M} = n < \infty$, let $\varphi_1, \dots \varphi_n$ be an orthonormal basis in $\mathcal{M} \ominus M_z\mathcal{M}$, and denote by ϕ the corresponding row operator-valued function. Then*

$$\mathcal{M} = \phi\mathcal{H}[\mathbf{C}], \tag{2}$$

where $\mathcal{H}[\mathbf{C}]$ consists of $\mathbb{C}^n$-valued functions and the mapping $g \mapsto \phi g$ is an isometry from $\mathcal{H}[\mathbf{C}]$ onto $\mathcal{M}$. Moreover,

$$\{\sum_{i=0}^{n} \varphi_i u_i : \; u_i \in \mathcal{A}, \; 1 \leq i \leq n\} \cap \mathcal{H} \tag{3}$$

is a dense subset of $\mathcal{M}$.

The dimension of $\mathcal{M} \ominus M_z \ominus$ is the *index* of the M_z-invariant subspace $\mathcal{M}$. We note that (2) continues to hold when the index is infinite. In fact, this dimension can be arbitrary even in the case of weighted shifts (see [16]). We do not know whether (3) holds in the infinite index case as well. Another natural question which arises is whether one can replace in (3) the disc algebra $\mathcal{A}$ by the set of polynomials. We will show that this can be done in the case that $\mathcal{H}[\mathbf{B}]$ has finite rank. For the infinite rank and index case we point to [26] where sufficient conditions are given under which (3) holds with $\mathcal{A}$ replaced by polynomials and $n = \infty$.

Theorem 1.1 can be used to investigate reverse Carleson measures on such spaces, a concept which has been studied in recent years in the context of $\mathcal{H}(b)$-spaces and model spaces (see [10], [19]). If $\mathcal{H} \cap \mathcal{A}$ is dense in $\mathcal{H}$, then a reverse Carleson measure for $\mathcal{H}$ is a measure μ on clos($\mathbb{D}$) such that $\|f\|^2_{\mathcal{H}} \leq C \int_{\text{clos}(\mathbb{D})} |f|^2 d\mu$ for $f \in \mathcal{H} \cap \mathcal{A}$. In Theorem 4.4 we prove that if the norm in $\mathcal{H}[\mathbf{B}]$ satisfies the identity

$$\|Lf\|^2 = \|f\|^2 - |f(0)|^2 \tag{4}$$

then the space cannot admit a reverse Carleson measure unless it is a backward shift invariant subspace of the Hardy space H^2. A class of spaces in which this identity holds has been studied in [25], where conditions are given on $\mathbf{B}$ which make the identity hold. It holds in all de Branges-Rovnyak spaces corresponding to extreme points of the unit ball of H^∞, so in particular our theorem answers a question in [10]. On the other hand if $\mathcal{H}$ is M_z-invariant, then reverse Carleson measures may exist and can be characterized in several ways. For example, in Theorem 4.2 we show that in this case $\mathcal{H} = \mathcal{H}[\mathbf{B}]$ admits a reverse Carleson measure if and only if $g := (1 - \sum_{i \in I} |b_i|^2)^{-1} \in L^1(\mathbb{T})$, and the measure gdm on $\mathbb{T}$ is essentially the minimal reverse Carleson measure for $\mathcal{H}$. The two conditions considered here yield almost a dichotomy. More precisely, if $\mathcal{H}[\mathbf{B}]$ satisfies (4) then it cannot be M_z-invariant unless it equals H^2.

Another application of Theorem 1.1 gives an approximation result for the orthogonal complements of M_z-invariant subspaces of the Bergman space $L^2_a(\mathbb{D})$. These might consist entirely of functions with bad integrability properties. For example, there are such subspaces $\mathcal{M}$ for which $\int_{\mathbb{D}} |f|^{2+\epsilon} dA = \infty$ holds for all $\epsilon > 0$ and $f \in \mathcal{M} \setminus \{0\}$ (see Proposition 4.7). Note that primitives of such Bergman space functions are not necessarily bounded in the disk. However, Corollary 4.8 below shows that the set of functions in $\mathcal{M}$ with a primitive in $\mathcal{A}$ is dense in $\mathcal{M}$.

The second part of the paper is devoted to the special case when $\mathcal{H}[\mathbf{B}]$ has finite rank, according to the definition above. Intuitively speaking, in this case the structure of the backward shift L resembles more a coisometry. The simplest examples are H^2 (rank zero) and the classical de Branges-Rovnyak spaces $\mathcal{H}(b)$ (rank one). Examples of higher rank $\mathcal{H}[\mathbf{B}]$-spaces are provided by Dirichlet-type spaces $\mathcal{D}(\mu)$ corresponding to measures μ with finite support in clos($\mathbb{D}$) (see [27], [3]).

It is not difficult to see that the rank of an $\mathcal{H}[\mathbf{B}]$-space is unstable with respect to equivalent Hilbert space norms. In fact, in [14] it is shown that $\mathcal{D}(\mu)$-spaces corresponding to a measure with finite support on $\mathbb{T}$ admit equivalent norms under which they become a rank one space. This leads to the fundamental question whether the (finite) rank of any $\mathcal{H}[\mathbf{B}]$-space can be reduced in this way. This question is addressed in Section 5.4. In Theorem 5.7 we relate the rank of $\mathcal{H}[\mathbf{B}]$ to the number of generators of a certain H^∞-submodule in the Smirnov class. In particular it turns out (Theorem 5.6) that there exist $\mathcal{H}[\mathbf{B}]$-spaces whose rank cannot be reduced by means of any equivalent norm satisfying (A.1)-(A.3).

In the case of finite rank $\mathcal{H}[\mathbf{B}]$-spaces our model becomes a very powerful tool. We use it in order to establish analogues of fundamental results from the theory of $\mathcal{H}(b)$-spaces. Moreover, we improve Corollary 1.2 to obtain a structure theorem for M_z-invariant subspaces which is new even in the rank one case. More explicitly, in Theorem 5.2 we show that a finite rank $\mathcal{H}[\mathbf{B}]$-space is M_z-invariant if and only if $\log(1 - \|\mathbf{B}\|_2^2)$ is integrable on $\mathbb{T}$. If this is the case, then we show in Theorem 5.5 that the polynomials are dense in the space. In Theorem 5.12 we turn to L-invariant subspaces and prove the analoque of a result of Sarason (see Theorem 5 of [29]), namely that every L-invariant subspace of $\mathcal{H}[\mathbf{B}]$ has the form $\mathcal{H}[\mathbf{B}] \cap K_\theta$, where $K_\theta = H^2 \ominus \theta H^2$ is a backward shift invariant subspaces of H^2. Concerning the structure of M_z-invariant subspaces we establish the following result.

Theorem 1.3. *Let $\mathcal{H} = \mathcal{H}[\mathbf{B}]$ be of finite rank and M_z-invariant. If $\mathcal{M}$ is a closed M_z-invariant subspace of $\mathcal{H}$, then* $\dim \mathcal{M} \ominus M_z\mathcal{M} = 1$ *and any non-zero element in $\mathcal{M} \ominus M_z\mathcal{M}$ is a cyclic vector for $M_z|\mathcal{M}$. Moreover, if $\phi \in \mathcal{M} \ominus M_z\mathcal{M}$ is of norm* 1*, then there exists a space $\mathcal{H}[\mathbf{C}]$ invariant under M_z, where $\mathbf{C} = (c_1, \ldots, c_k)$ and $k \leq n$, such that*

$$\mathcal{M} = \phi\mathcal{H}[\mathbf{C}]$$

and the mapping $g \mapsto \phi g$ is an isometry from $\mathcal{H}[\mathbf{C}]$ onto $\mathcal{M}$.

This result follows from Theorem 5.11 below. In fact, the mentioned theorem establishes a more precise description of the invariant subspace in terms of our model (see part (iv) of Theorem 5.11).

2 Basic structure

Throughout the paper, vectors and vector-valued functions will usually be denoted by boldface letters like $\mathbf{c}$, $\mathbf{f}$, while operators, matrices and operator-valued functions will usually be denoted by capitalized boldface letters like $\mathbf{B}$, $\mathbf{A}$. The space of bounded linear operators between two Hilbert spaces X, Y will be denoted by $\mathcal{B}(Y, X)$, and we simply write $\mathcal{B}(X)$ if $X = Y$. All appearing Hilbert spaces will be assumed separable. The identity operator on a space X will be denoted by I_X. The backward shift operation will always be denoted by L, regardless of the space it acts upon, and regardless of if the operand is a scalar-valued function or a vector-valued function. The same conventions will be used for the forward shift operator M_z. If Y is a Hilbert space, then we denote by $H^2(Y)$ the Hardy space of analytic functions $\mathbf{f} : \mathbb{D} \to Y$ with square-summable Taylor coefficients, and $H^\infty(Y)$ is the space of bounded analytic functions from $\mathbb{D}$ to Y. The concepts of *inner* and *outer* functions are defined as usual (see Chapter V of [31]). The space $L^2(Y)$ will denote the space of square-integrable Y-valued measurable functions defined on the circle $\mathbb{T}$, and we will identify $H^2(Y)$ as a closed subspace of $L^2(Y)$ in the usual manner by considering the boundary values of the analytic functions $\mathbf{f} \in H^2(Y)$. In the case $Y = \mathbb{C}$ we will simply write H^2 and L^2. The orthogonal complement of $H^2(Y)$ inside $L^2(Y)$ will be denoted by $\overline{H^2_0(Y)}$. The norm in $L^2(Y)$ and its subspaces will be denoted by $\| \cdot \|_2$. The inner product of two elements f, g in a Hilbert space $\mathcal{H}$ will be denoted by $\langle f, g \rangle_{\mathcal{H}}$.

2.1 Reproducing kernel. As mentioned in the introduction, we will actually work in the context of vector-valued analytic functions, which will be necessary in order to prove Theorem 1.1 in full generality. Thus, let $X, \mathcal{H}$ be Hilbert spaces, where $\mathcal{H}$ consists of analytic functions $\mathbf{f} : \mathbb{D} \to X$. The versions of axioms (A.1)-(A.3) in the X-valued context are:

(A.1') The evaluation $\mathbf{f} \mapsto \langle \mathbf{f}(\lambda), x \rangle_X$ is a bounded linear functional on $\mathcal{H}$ for each $\lambda \in \mathbb{D}$ and $x \in X$,

(A.2') $\mathcal{H}$ is invariant under the backward shift operator L given by

$$L\mathbf{f}(z) = \frac{\mathbf{f}(z) - \mathbf{f}(0)}{z},$$

and we have that

$$\|L\mathbf{f}\|_{\mathcal{H}} \leq \|\mathbf{f}\|_{\mathcal{H}}, \quad \mathbf{f} \in \mathcal{H},$$

(A.3') the constant vectors $x \in X$ are contained in $\mathcal{H}$ and have the reproducing property

$$\langle \mathbf{f}, x \rangle_{\mathcal{H}} = \langle \mathbf{f}(0), x \rangle_X \quad \mathbf{f} \in \mathcal{H}, x \in X.$$

By (A.1') there exists an operator-valued reproducing kernel $k_{\mathcal{H}} : \mathbb{D} \times \mathbb{D} \to \mathcal{B}(X)$ such that for each $\lambda \in \mathbb{D}$ and $x \in X$ the identity

$$\langle \mathbf{f}, k_{\mathcal{H}}(\cdot, \lambda)x \rangle_{\mathcal{H}} = \langle \mathbf{f}(\lambda), x \rangle_X$$

holds for $\mathbf{f} \in \mathcal{H}$. Axiom (A.3') implies that $k_{\mathcal{H}}(z, 0) = I_X$ for each $z \in \mathbb{D}$.

Proposition 2.1. *Let $\mathcal{H}$ be a Hilbert space of analytic functions which satisfies the axioms (A.1')-(A.3'). Then there exists a Hilbert space Y and an analytic function $\mathbf{B} : \mathbb{D} \to \mathcal{B}(Y, X)$ such that for each $z \in \mathbb{D}$ the operator $\mathbf{B}(z) : Y \to X$ is a contraction, and*

$$k_{\mathcal{H}}(z, \lambda) = \frac{I_X - \mathbf{B}(z)\mathbf{B}(\lambda)^*}{1 - \overline{\lambda} z}. \tag{5}$$

In particular, if $X = \mathbb{C}$, then $\mathbf{B}(z)$ is an analytic row contraction into l^2, i.e. there exist analytic functions $\{b_i\}_{i\geq 1}$ in $\mathbb{D}$ such that

$$k_{\mathcal{H}}(\lambda, z) = \frac{1 - \sum_{i\geq 1} b_i(z)\overline{b_i(\lambda)}}{1 - \overline{\lambda} z}, \quad \sum_{i\geq 1} |b_i(z)|^2 \leq 1,\ z \in \mathbb{D}.$$

If $\dim(I_{\mathcal{H}} - LL^*)\mathcal{H} = n < \infty$, then there exists a representation as above, with $b_i = 0,\ i > n$, and such that the functions $b_1, \ldots, b_n$ are linearly independent.

Proof. Let $k = k_{\mathcal{H}}$ be the reproducing kernel of $\mathcal{H}$. Fix a vector $x \in X$ and $\lambda \in \mathbb{D}$. By (A.3') we have

$$\begin{aligned} \langle \mathbf{f}, (k(\cdot, \lambda) - I_X)x/\overline{\lambda} \rangle_{\mathcal{H}} &= \langle (\mathbf{f}(\lambda) - \mathbf{f}(0))/\lambda, x \rangle_X = \langle L\mathbf{f}, k(\cdot, \lambda)x \rangle_{\mathcal{H}} \\ &= \langle \mathbf{f}, L^* k(\cdot, \lambda)x \rangle_{\mathcal{H}}. \end{aligned}$$

It follows that

$$L^* k(z, \lambda)x = \frac{(k(z, \lambda) - I_X)x}{\overline{\lambda}}$$

and

$$(I_{\mathcal{H}} - LL^*)k(z, \lambda)x = k(z, \lambda)x - \frac{(k(z, \lambda) - I_X)x}{\overline{\lambda} z}. \tag{6}$$

By (A.2') the operator $P := I_{\mathcal{H}} - LL^*$ is positive. Therefore $Pk(\cdot, \lambda)$ is a positive-definite kernel and hence it has a factorization $Pk(z, \lambda) = \tilde{\mathbf{B}}(z)\tilde{\mathbf{B}}(\lambda)^*$ for some operator-valued analytic function $\tilde{\mathbf{B}} : \mathbb{D} \to \mathcal{B}(Y, X)$ (see Chapter 2 of [1]). We can now solve for k in (6) to obtain

$$k(z, \lambda) = \frac{I_X - \mathbf{B}(z)\mathbf{B}(\lambda)^*}{1 - \overline{\lambda} z},$$

where we have set $\mathbf{B}(z) := z\tilde{\mathbf{B}}(z)$. It is clear from this expression and the positive-definiteness of k that $\mathbf{B}(z)$ must be a contraction for every $z \in \mathbb{D}$. If $\dim(I_{\mathcal{H}} - LL^*) < \infty$, then the last assertion of the proposition follows in a standard manner from (6) and the spectral theorem applied to the finite rank operator $I_{\mathcal{H}} - LL^*$. $\square$

2.2 Model. The space $\mathcal{H}$ is completely determined by the function $\mathbf{B} : \mathbb{D} \to \mathcal{B}(Y, X)$ appearing in Proposition 2.1. To emphasize this we will on occasion write $\mathcal{H}[\mathbf{B}]$ in place of $\mathcal{H}$. The function $\mathbf{B}$ admits a non-tangential boundary value $\mathbf{B}(\zeta)$ for almost every $\zeta \in \mathbb{T}$ (convergence in the sense of strong operator topology), and the operator

$$\Delta(\zeta) = (I_{\mathcal{H}} - \mathbf{B}(\zeta)^*\mathbf{B}(\zeta))^{1/2} \tag{7}$$

induces in a natural way a multiplication operator from $H^2(Y)$ to $L^2(Y)$. The space $\mathrm{clos}(\Delta H^2(Y))$ is a subspace of $L^2(Y)$ which is invariant under the operator M_ζ given by $M_\zeta \mathbf{g}(\zeta) = \zeta \mathbf{g}(\zeta), \zeta \in \mathbb{T}$. This implies that it can be decomposed as

$$\mathrm{clos}(\Delta H^2(Y)) = W \oplus \Theta H^2(Y_1) \tag{8}$$

where M_ζ acts unitarily on W, Y_1 is an auxilliary Hilbert space, $\Theta : \mathbb{T} \to \mathcal{B}(Y_1, Y)$ is a measurable operator-valued function such that for almost every $\zeta \in \mathbb{T}$ the operator $\Theta(\zeta) : Y_1 \to Y$ is isometric, and the functions in W and $\Theta H^2(Y_1)$ are pointwise orthogonal almost everywhere, i.e, $\langle \mathbf{f}(\zeta), \mathbf{g}(\zeta) \rangle_Y = 0$ for almost every $\zeta \in \mathbb{T}$ for any pair $\mathbf{f} \in W$ and $\mathbf{g} \in \Theta H^2(Y_1)$ (see Theorem 9 of Lecture VI in [22]). It follows that $\Theta(\zeta)^*\mathbf{f}(\zeta) = 0$ for all $\mathbf{f} \in W$, and consequently

$$\Theta^*\mathrm{clos}(\Delta H^2(Y)) = \Theta^* W \oplus \Theta^*\Theta H^2(Y_1) = \{0\} \oplus H^2(Y_1).$$

The above computation shows that $\Theta^*\Delta$ maps $H^2(Y)$ to a dense subset of $H^2(Y_1)$, and so standard theory of operator-valued functions implies that there exists an analytic outer function $\mathbf{A} : \mathbb{D} \to \mathcal{B}(Y, Y_1)$ such that $\mathbf{A}(\zeta) = \Theta^*(\zeta)\Delta(\zeta)$ for almost every $\zeta \in \mathbb{T}$.

Theorem 2.2. *There exists an isometric embedding $J : \mathcal{H}[\mathbf{B}] \to H^2(X) \oplus \mathrm{clos}(\Delta H^2(Y))$ satisfying the following properties.*

(i) A function $\mathbf{f} \in H^2(X)$ is contained in $\mathcal{H}[\mathbf{B}]$ if and only if there exists $\mathbf{g} \in \mathrm{clos}(\Delta H^2(Y))$ such that

$$\mathbf{B}^*\mathbf{f} + \Delta\mathbf{g} \in \overline{H^2_0(Y)}.$$

If this is the case, then $\mathbf{g}$ is unique and

$$J\mathbf{f} = (\mathbf{f}, \mathbf{g}).$$

(ii) If $J\mathbf{f} = (\mathbf{f}, \mathbf{g})$ and $\mathbf{g} = \mathbf{w} + \Theta\mathbf{f}_1$ is the decomposition of $\mathbf{g}$ with respect to (8)*, then*

$$JL\mathbf{f} = (L\mathbf{f}, \overline{\zeta}\mathbf{w} + \Theta L\mathbf{f}_1).$$

(iii) The orthogonal complement of $J\mathcal{H}[\mathbf{B}]$ inside $H^2(X) \oplus \mathrm{clos}(\Delta H^2(Y))$ is

$$(J\mathcal{H}[\mathbf{B}])^\perp = \{(\mathbf{B}\mathbf{h}, \Delta\mathbf{h}) : \mathbf{h} \in H^2(Y)\}.$$

Proof. Let $K = H^2(X) \oplus \text{clos}(\Delta H^2(Y)), U = \{(\mathbf{B}\mathbf{h}, \Delta\mathbf{h}) : \mathbf{h} \in H^2(Y)\}$ and note that U is a closed subspace of K. Let P be the projection taking a tuple $(\mathbf{f}, \mathbf{g}) \in K$ to $\mathbf{f} \in H^2(X)$. Then P is injective on $K \ominus U$. Indeed, if $(0, \mathbf{g})$ is orthogonal to U, then $\mathbf{g} \in \text{clos}(\Delta H^2(Y))$ is orthogonal to $\Delta H^2(Y)$, and hence $\mathbf{g} = 0$. Any analytic function $\mathbf{f}$ can thus appear in at most one tuple $(\mathbf{f}, \mathbf{g}) \in K \ominus U$. We define $\mathcal{H}_0 = P(K \ominus U)$ as the space of analytic X-valued functions with the norm

$$\|\mathbf{f}\|^2_{\mathcal{H}_0} := \|\mathbf{f}\|_2^2 + \|\mathbf{g}\|_2^2.$$

We will see that $\mathcal{H}[\mathbf{B}] = \mathcal{H}_0$ by showing that the reproducing kernels of the two spaces are equal. Note that for each $c \in X$ the tuple

$$k_{c,\lambda} = \left(\frac{(I_X - \mathbf{B}(z)\mathbf{B}(\lambda)^*)c}{1 - \overline{\lambda} z}, \frac{-\Delta(\zeta)\mathbf{B}(\lambda)^* c}{1 - \overline{\lambda} z} \right) \in K$$

is orthogonal to U, and therefore its first component $Pk_{c,\lambda}$ defines an element of $\mathcal{H}_0$. Moreover, it follows readily from our definitions that for any $\mathbf{f} \in \mathcal{H}_0$ we have

$$\langle \mathbf{f}, Pk_{c,\lambda} \rangle_{\mathcal{H}_0} = \langle \mathbf{f}(\lambda), c \rangle_X.$$

Thus the reproducing kernel of $\mathcal{H}_0$ equals the one given by (5), and so $\mathcal{H}[\mathbf{B}] = \mathcal{H}_0$.

It is clear from the above paragraph that a function $\mathbf{f} \in H^2(X)$ is contained in $\mathcal{H}[\mathbf{B}]$ if and only if there exists $\mathbf{g} \in \text{clos}(\Delta H^2(Y))$ such that $(\mathbf{f}, \mathbf{g})$ is orthogonal to $(\mathbf{B}\mathbf{h}, \Delta\mathbf{h})$ for all $\mathbf{h} \in H^2(Y)$, that is to say, if and only if $\mathbf{B}(\zeta)^*\mathbf{f}(\zeta) + \Delta(\zeta)\mathbf{g}(\zeta) \in \overline{H^2_0(Y)}$. If we let $J = P^{-1}$, then $J\mathbf{f} = (\mathbf{f}, \mathbf{g})$, and part (i) follows. Part (iii) holds by construction. In order to prove (ii), it will be sufficient by (i) to show that

$$\mathbf{B}^* L\mathbf{f} + \Delta(\overline{\zeta}\mathbf{w} + \Theta L\mathbf{f}_1) \in \overline{H^2_0(Y)}. \tag{9}$$

Let $\mathbf{A} = \Theta^*\Delta$ be the analytic function mentioned above. We have that

$$\overline{\zeta}(\mathbf{B}^*\mathbf{f} + \Delta\mathbf{g}) = \overline{\zeta}\mathbf{B}^*\mathbf{f} + \Delta\overline{\zeta}\mathbf{w} + \overline{\zeta}\mathbf{A}^*\mathbf{f}_1 \in \overline{H^2_0(Y)}. \tag{10}$$

The term $\mathbf{A}^* L\mathbf{f}_1$ differs from $\overline{\zeta}\mathbf{A}^*\mathbf{f}_1$ only by a function in $\overline{H^2_0(Y)}$, and the same is true for $\mathbf{B}^* L\mathbf{f}$ and $\overline{\zeta}\mathbf{B}^*\mathbf{f}$. Thus (9) follows from (10). □

2.3 Analytic model. The model of Theorem 2.2 can be greatly simplifed if $W = \{0\}$ in the decomposition (8). The condition for when this occurs can be expressed in terms of L.

Corollary 2.3. *We have* $W = \{0\}$ *in* (8) *if and only if* $\|L^n\mathbf{f}\|_{\mathcal{H}[\mathbf{B}]} \to 0$ *as* $n \to \infty$, *for all* $f \in \mathcal{H}[\mathbf{B}]$.

Proof. Assume first that $W = \{0\}$. If $J\mathbf{f} = (\mathbf{f}, \Theta\mathbf{h})$, then by (ii) of Theorem 2.2 we have that

$$\|L^n\mathbf{f}\|^2_{\mathcal{H}[\mathbf{B}]} = \|L^n\mathbf{f}\|^2_2 + \|L^n\mathbf{h}\|^2_2$$

which clearly tends to 0 as $n \to \infty$.

Conversely, note that the convergence of $L^n\mathbf{f}$ to zero implies that if $J\mathbf{f} = (\mathbf{f}, \mathbf{g})$ and $\mathbf{g} = \mathbf{w} + \Theta\mathbf{h}$ is the decomposition of $\mathbf{g}$ with respect to (8), then $\mathbf{w} = 0$ (else $\lim_{n\to\infty} \|L^n\mathbf{f}\|_{\mathcal{H}[\mathbf{B}]} \geq \|\mathbf{w}\|_2$ by (ii) of Theorem 2.2). Thus for any $\mathbf{w} \in W$ we have $J\mathbf{f} \perp \mathbf{w}$ for all $\mathbf{f} \in \mathcal{H}[\mathbf{B}]$, and so $(0, \mathbf{w}) = (\mathbf{B}\mathbf{h}, \Delta\mathbf{h})$ for some $\mathbf{h} \in H^2(Y)$ by Theorem 2.2. Since $\mathbf{B}\mathbf{h} = 0$, we deduce that $\mathbf{w} = \Delta\mathbf{h} = \mathbf{h}$, for example by approximating the function $x \mapsto \sqrt{1-x}$ uniformly on $[0,1]$ by a sequence of polynomials p_n with $p_n(0) = 1$, so that $\Delta\mathbf{h} = \lim_{n\to\infty} p_n(\mathbf{B}^*\mathbf{B})\mathbf{h} = \mathbf{h}$. It follows that $\mathbf{w} = \mathbf{h} \in H^2(Y)$. Since $\mathbf{w}$ was arbitrary, we deduce that $W \subset H^2(Y)$, and since M_ζ acts unitarily on W, we must have $W = \{0\}$. □

Assume we are in the case described by Corollary 2.3. Then (8) reduces to

$$\text{clos}(\Delta H^2(Y)) = \Theta H^2(Y_1).$$

Thus it holds that $\text{clos}(\text{Im}\, \Delta(\zeta)) = \text{Im}\, \Theta(\zeta)$ for almost every $\zeta \in \mathbb{T}$. Since $\Theta(\zeta)\Theta(\zeta)^*$ is equal almost everywhere to the projection of Y onto $\text{Im}\, \Theta(\zeta)$, we obtain

$$\mathbf{A}^*(\zeta)\mathbf{A}(\zeta) = \Delta(\zeta)\Theta(\zeta)\Theta(\zeta)^*\Delta(\zeta) = \Delta(\zeta)^2$$

and consequently $\mathbf{B}(\zeta)^*\mathbf{B}(\zeta) + \mathbf{A}(\zeta)^*\mathbf{A}(\zeta) = 1_Y$ for almost every $\zeta \in \mathbb{T}$. The following theorem is a reformulation of Theorem 2.2. We omit the proof, which can be easily deduced from the proof of Theorem 2.2 with $\mathbf{A}$ playing the role of Δ.

Theorem 2.4. *Let $\mathcal{H}[\mathbf{B}]$ be such that $\|L^n\mathbf{f}\|_{\mathcal{H}[\mathbf{B}]} \to 0$ as $n \to \infty$, for all $f \in \mathcal{H}[\mathbf{B}]$. There exists an auxilliary Hilbert space Y_1, an outer function $\mathbf{A} : \mathbb{D} \to \mathcal{B}(Y, Y_1)$ such that*

$$\mathbf{B}(\zeta)^*\mathbf{B}(\zeta) + \mathbf{A}(\zeta)^*\mathbf{A}(\zeta) = 1_Y$$

for almost every $\zeta \in \mathbb{T}$, and an isometric embedding $J : \mathcal{H}[\mathbf{B}] \to H^2(X) \oplus H^2(Y_1)$ satisfying the following properties.

(i) A function $\mathbf{f} \in H^2(X)$ is contained in $\mathcal{H}[\mathbf{B}]$ if and only if there exists $\mathbf{f}_1 \in H^2(Y_1)$ such that

$$\mathbf{B}^*\mathbf{f} + \mathbf{A}^*\mathbf{f}_1 \in \overline{H^2_0(Y)}.$$

If this is the case, then $\mathbf{f}_1$ is unique, and

$$J\mathbf{f} = (\mathbf{f}, \mathbf{f}_1).$$

(ii) If $J\mathbf{f} = (\mathbf{f}, \mathbf{f}_1)$, then

$$J\mathbf{f} = (L\mathbf{f}, L\mathbf{f}_1).$$

(iii) The orthogonal complement of $J\mathcal{H}[\mathbf{B}]$ inside $H^2(X) \oplus H^2(Y_1)$ is

$$(\mathcal{H}[\mathbf{B}])^\perp = \{(\mathbf{B}\mathbf{h}, \mathbf{A}\mathbf{h}) : \mathbf{h} \in H^2(Y)\}.$$

2.4 **A formula for the norm.** As pointed out in the introduction, the model of Theorem 2.2 has a connection to a useful formula for the norm in Hilbert spaces of analytic functions.

Proposition 2.5. *Let $\mathcal{H}$ be a Hilbert space of analytic functions which satisfies (A.1')-(A.3'). For $\mathbf{f} \in \mathcal{H}$ we have*

$$\|\mathbf{f}\|_{\mathcal{H}}^2 = \|\mathbf{f}\|_2^2 + \lim_{r\to 1} \int_{\mathbb{T}} \Big\| z \frac{\mathbf{f}(z) - \mathbf{f}(r\lambda)}{z - r\lambda} \Big\|_{\mathcal{H}}^2 - r^2 \Big\| \frac{\mathbf{f}(z) - \mathbf{f}(r\lambda)}{z - r\lambda} \Big\|_{\mathcal{H}}^2 dm(\lambda). \tag{11}$$

Note that the formula makes sense even if $\mathcal{H}$ is not invariant for M_z, since for $\lambda \in \mathbb{D}$ we have

$$z\frac{\mathbf{f}(z) - \mathbf{f}(\lambda)}{z - \lambda} = \frac{z\mathbf{f}(z) - \lambda \mathbf{f}(\lambda)}{z - \lambda} - \mathbf{f}(\lambda) = (1 - \lambda L)^{-1}\mathbf{f}(z) - \mathbf{f}(\lambda) \in \mathcal{H},$$

and $(1 - \lambda L)^{-1}$ exists since L is a contraction on $\mathcal{H}$. Versions of the above formula have been used in a crucial way in several works related to the structure of invariant subspaces, see for example [5], [7] and [8]. We shall prove the formula by verifying that if $J\mathbf{f} = (\mathbf{f}, \mathbf{g})$ in Theorem 2.2, then the limit in (11) is equal to $\|\mathbf{g}\|_2^2$. Actually, we will prove a stronger result than Proposition 2.5, one which we will find useful at a later stage. In the next theorem and in the sequel we use the notation

$$L_\lambda := L(1 - \lambda L)^{-1}, \lambda \in \mathbb{D}.$$

One readily verifies that

$$L_\lambda \mathbf{f}(z) = \frac{\mathbf{f}(z) - \mathbf{f}(\lambda)}{z - \lambda}.$$

Theorem 2.6. *Let $J\mathbf{f} = (\mathbf{f}, \mathbf{w} + \Theta \mathbf{f}_1)$ as in Theorem 2.2, or $J\mathbf{f} = (\mathbf{f}, \mathbf{f}_1)$ as in Theorem 2.4. In both cases, we have that*

(i) $\|zL_\lambda \mathbf{f}\|_{\mathcal{H}[\mathbf{B}]}^2 - \|L_\lambda \mathbf{f}\|_{\mathcal{H}[\mathbf{B}]}^2 = \|\mathbf{f}_1(\lambda)\|_{Y_1}^2$,

(ii) $\|\mathbf{f}_1\|_2^2 = \lim_{r\to 1} \int_{\mathbb{T}} \|zL_{r\lambda}\mathbf{f}\|_{\mathcal{H}[\mathbf{B}]}^2 - \|L_{r\lambda}\mathbf{f}\|_{\mathcal{H}[\mathbf{B}]}^2 dm(\lambda)$,

(iii) $\|\mathbf{w}\|_2^2 = \lim_{r\to 1} \int_{\mathbb{T}} (1 - r^2) \|L_{r\lambda}\mathbf{f}\|_{\mathcal{H}[\mathbf{B}]}^2 dm(\lambda)$.

In particular, (11) *holds.*

Proof. We shall only carry out the proof in the context of Theorem 2.2, the other case being similar. First, we claim that for $\lambda \in \mathbb{D}$ we have that

$$JzL_\lambda \mathbf{f} = \big(zL_\lambda \mathbf{f}, (1 - \lambda\overline{\zeta})^{-1}\mathbf{w} + \Theta(zL_\lambda \mathbf{f}_1 + \mathbf{f}_1(\lambda))\big). \tag{12}$$

This follows from part (i) of Theorem 2.2. Indeed, we have to check that

$$\begin{aligned}\mathbf{B}(\zeta)^*\zeta\frac{\mathbf{f}(\zeta)-\mathbf{f}(\lambda)}{\zeta-\lambda}+\Delta(\zeta)\frac{\mathbf{w}(\zeta)}{1-\lambda\overline{\zeta}}+\Delta(\zeta)\Theta(\zeta)\zeta\frac{\mathbf{f}_1(\zeta)-\mathbf{f}_1(\lambda)}{\zeta-\lambda}+\Delta(\zeta)\Theta(\zeta)\mathbf{f}_1(\lambda)\\ =\frac{\mathbf{B}(\zeta)^*\mathbf{f}(\zeta)+\Delta(\zeta)\mathbf{g}(\zeta)}{1-\lambda\overline{\zeta}}-\frac{\mathbf{B}(\zeta)^*\mathbf{f}(\lambda)}{1-\lambda\overline{\zeta}}-\Big(\frac{\mathbf{A}(\zeta)^*\mathbf{f}_1(\lambda)}{1-\lambda\overline{\zeta}}-\mathbf{A}(\zeta)^*\mathbf{f}_1(\lambda)\Big)\end{aligned}$$

lies in $\overline{H^2_0(Y)}$, and this is true since each of the three terms in the last line lies in $\overline{H^2_0(Y)}$. Similarly, we have

$$JL_\lambda\mathbf{f}=\big(L_\lambda\mathbf{f},\overline{\zeta}(1-\lambda\overline{\zeta})^{-1}\mathbf{w}+\Theta L_\lambda\mathbf{f}_1\big).$$

Actually, this can be seen immediately by applying (ii) of Theorem 2.2 to (12). Since J is an isometry we have that

$$\|zL_\lambda\mathbf{f}\|^2_{\mathcal{H}[\mathbf{B}]}=\|zL_\lambda\mathbf{f}\|^2_2+\|(1-\lambda\overline{\zeta})^{-1}\mathbf{w}\|^2_2+\|zL_\lambda\mathbf{f}_1+\mathbf{f}_1(\lambda)\|^2_2 \tag{13}$$

and

$$\|L_\lambda\mathbf{f}\|^2_{\mathcal{H}[\mathbf{B}]}=\|L_\lambda\mathbf{f}\|^2_2+\|\overline{\zeta}(1-\lambda\overline{\zeta})^{-1}\mathbf{w}\|^2_2+\|L_\lambda\mathbf{f}_1\|^2_2. \tag{14}$$

The difference of (13) and (14) equals $\|\mathbf{f}_1(\lambda)\|^2_{Y_1}$ which gives (i). Part (ii) is immediate from (i). We will deduce part (iii) from (14). A brief computation involving power series shows that if T is a contraction on a Hilbert space $\mathcal{H}$, then we have $\|T^nx\|_{\mathcal{H}}\to 0$ if and only if

$$\lim_{r\to1^-}\int_{\mathbb{T}}(1-r^2)\|T(1-r\lambda T)^{-1}x\|^2_{\mathcal{H}}dm(\lambda)=0.$$

We apply this to the $T=L$ acting on the Hardy spaces, and deduce (iii) from (14):

$$\begin{aligned}\lim_{r\to1}\int_{\mathbb{T}}(1-r^2)\|L_{r\lambda}\mathbf{f}\|^2_{\mathcal{H}[\mathbf{B}]}dm(\lambda)&=\lim_{r\to1}\int_{\mathbb{T}}(1-r^2)\|(1-r\lambda\overline{\zeta})^{-1}\mathbf{w}\|^2_2dm(\lambda)\\ =\lim_{r\to1}\int_{\mathbb{T}}\left(\int_{\mathbb{T}}(1-r^2)|1-r\lambda\overline{\zeta}|^{-2}dm(\lambda)\right)&\|\mathbf{w}(\zeta)\|^2_Ydm(\zeta)=\|\mathbf{w}\|^2_2.\end{aligned}$$

□

2.5 Examples. We end this section by discussing some examples of spaces which satisfy our assumptions, some of which were already mentioned in the introduction.

2.5.1 De Branges-Rovnyak spaces. If $\mathbf{B}=b$ is a non-zero scalar-valued function, then $\Delta=(1-|b|^2)^{1/2}$ is an operator on a 1-dimensional space, and clos$(\Delta H^2)\subseteq L^2$ is either of the form θH^2 for some unimodular function θ on $\mathbb{T}$, or it is of the form $L^2(E):=\{f\in L^2: f\equiv 0$ a.e. on $\mathbb{T}\setminus E\}$. The first case corresponds to b which are non-extreme points of the unit ball of H^∞, while the second case corresponds to the extreme points. It is in the first case that the model of Theorem 2.4 applies to $\mathcal{H}(b)$.

2.5.2 Weighted H^2-spaces. Let $w = (w_n)_{n\geq 0}$ be a sequence of positive numbers and H^2_w be the space of analytic functions in $\mathbb{D}$ which satisfy

$$\|f\|^2_{H^2_w} := \sum_{n=0}^{\infty} w_n |f_n|^2 < \infty,$$

where f_n is the nth Taylor coefficient of f at $z = 0$. If $w_0 = 1$ and $w_{n+1} \geq w_n$ for $n \geq 0$, then H^2_w satisfies (A.1)-(A.3). The reproducing kernel of H^2_w is given by

$$k(z, \lambda) = \sum_{n=0}^{\infty} \frac{\overline{\lambda^n} z^n}{w_n} = \frac{1 - \sum_{n=1}^{\infty}(1/w_{n-1} - 1/w_n)\overline{\lambda^n} z^n}{1 - \overline{\lambda} z},$$

and thus $H^2_w = \mathcal{H}[\mathbf{B}]$ with $\mathbf{B}(z) = \big(\sqrt{1/w_{n-1} - 1/w_n} z^n\big)_{n=1}^{\infty}$.

2.5.3 Dirichlet-type spaces. Let μ be a positive finite Borel measure on $\mathrm{clos}(\mathbb{D})$. The space $\mathcal{D}(\mu)$ consists of analytic functions which satisfy

$$\|f\|^2_{\mathcal{D}(\mu)} := \|f\|_2^2 + \int_{\mathbb{T}} \int_{\mathrm{clos}(\mathbb{D})} \frac{|f(z) - f(\zeta)|^2}{|z - \zeta|^2} d\mu(z) dm(\zeta).$$

The choice of $d\mu = dm$ produces the classical Dirichlet space. It is easy to verify directly from this expression that $\mathcal{D}(\mu)$ satisfies axioms (A.1)-(A.3), but it is in general difficult to find an expression for $\mathbf{B}$ corresponding to the space as in Proposition 2.1. In the special case that $\mu = \sum_{i=1}^{n} c_i \delta_{z_i}$ is a positive sum of unit masses δ_{z_i} at distinct points $z_i \in \mathrm{clos}(\mathbb{D})$ the space $\mathcal{D}(\mu)$ is an $\mathcal{H}[\mathbf{B}]$-space of rank n. An isometric embedding $J : \mathcal{D}(\mu) \to (H^2)^{n+1}$ satisfying the properties listed in Theorem 2.4 is given by

$$f(z) \mapsto \Big(f(z), \frac{f(z) - f(z_1)}{z - z_1}, \ldots, \frac{f(z) - f(z_n)}{z - z_n}\Big).$$

2.5.4 Cauchy duals. Let $\mathcal{H}$ be a Hilbert space of analytic functions which contains all functions holomorphic in a neighbourhood of $\mathrm{clos}(\mathbb{D})$, on which the forward shift operator M_z acts as a contraction and such that $\langle f, 1 \rangle_{\mathcal{H}} = f(0)$ holds for $f \in \mathcal{H}$. Consider the function

$$Uf(\lambda) = \big\langle (1 - \lambda z)^{-1}, f(z) \big\rangle_{\mathcal{H}} = \sum_{n=0}^{\infty} \lambda^n \big\langle z^n, f(z) \big\rangle_{\mathcal{H}}.$$

Then Uf is an analytic function of λ, and $UM_z^* f = LUf$. If $\mathcal{H}^*$ is defined to be the space of functions of the form Uf for $f \in \mathcal{H}$, with the norm $\|Uf\|_{\mathcal{H}^*} = \|f\|_{\mathcal{H}}$, then it is easy to verify that $\mathcal{H}^*$ is a Hilbert space of analytic functions which satisfies (A.1)-(A.3). The space $\mathcal{H}^*$ is the so-called Cauchy dual of $\mathcal{H}$ (see [4]).

3 Density of functions with continuous extensions to the closed disk

The goal of this section is to prove Theorem 3.5 and Corollary 3.6, which are vector-valued generalizations of Theorem 1.1 mentioned in the introduction. We will first recall a few facts about the disk algebra $\mathcal{A}$ and the vector-valued Smirnov classes $N^+(Y)$.

3.1 Disk algebra, Cauchy transforms and the Smirnov class. Let $\mathcal{A}$ denote the disk algebra, the space of scalar-valued analytic functions defined in $\mathbb{D}$ which admit continuous extensions to clos($\mathbb{D}$). It is a Banach space if given the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$, and the dual of $\mathcal{A}$ can be identified with the space $\mathcal{C}$ of Cauchy transforms of finite Borel measures μ supported on the circle $\mathbb{T}$. A Cauchy transform f is an analytic function in $\mathbb{D}$ which is of the form $f(z) = C\mu(z) := \int_\mathbb{T} \frac{1}{1-z\overline{\zeta}} d\mu(\zeta)$ for some Borel measure μ. The duality between $\mathcal{A}$ and $\mathcal{C}$ is realized by

$$\langle h, f\rangle = \lim_{r\to 1^-} \int_\mathbb{T} h(r\zeta)\overline{f(r\zeta)} dm(\zeta) = \int_\mathbb{T} h \overline{d\mu}, \quad h \in \mathcal{A}, f = C\mu$$

and the norm $\|f\|_\mathcal{C}$ of f as a functional on $\mathcal{A}$ is given by $\|f\|_\mathcal{C} = \inf_{\mu: C\mu = f} \|\mu\|$, where $\|\mu\|$ is the total variation of the measure μ. The space $\mathcal{C}$ is continuously embedded in the Hardy space H^p for each $p \in (0,1)$. More precisely, we have for each fixed $p \in (0,1)$ the estimate $\|f\|_p = (\int_\mathbb{T} |f|^p dm)^{1/p} \leq c_p \|f\|_\mathcal{C}$. As a dual space of $\mathcal{A}$, the space $\mathcal{C}$ can be equipped with the weak-star topology, and a sequence (f_n) converges weak-star to $f \in \mathcal{C}$ if and only if $\sup_n \|f_n\|_\mathcal{C} < \infty$ and $f_n(z) \to f(z)$ for each $z \in \mathbb{D}$. See [13] for more details.

If Y is a Hilbert space, then the Smirnov class $N^+(Y)$ consists of the functions $\mathbf{f} : \mathbb{D} \to Y$ which can be written as $\mathbf{f} = \mathbf{u}/v$, where $\mathbf{u} \in H^\infty(Y)$ and $v : \mathbb{D} \to \mathbb{C}$ is a bounded outer function. In the case $Y = \mathbb{C}$ we will simply write N^+. The class $N^+(Y)$ satisfies the following *Smirnov maximum principle*: if $\mathbf{f} \in N^+(Y)$, then we have that $\int_\mathbb{T} \|\mathbf{f}(\zeta)\|_Y^2 dm(\zeta) < \infty$ if and only if $\mathbf{f} \in H^2(Y)$ (see Theorem A in Section 4.7 of [28]).

3.2 Proof of the density theorem. The proof will depend on a series of lemmas. The first two are routine exercises in functional analysis and the proofs of those will be omitted.

Lemma 3.1. *Let B be a Banach space, B' be its dual space, and $S \subset B'$ be a linear manifold. If $l \in B'$ annihilates the subspace $\cap_{s\in S} \ker s \subset B$, then l lies in the weak-star closure of S.*

Lemma 3.2. *Let $\{h_j\}$ be a sequence of scalar-valued analytic functions in $\mathbb{D}$, with $\sup_n \|h_n\|_\infty < \infty$, and which converges uniformly on compacts to the function h. If the sequence $\{\mathbf{g}_j\}_{j=1}^\infty$ of functions in $L^2(Y)$ converges in norm to $\mathbf{g}$, then $h_j\mathbf{g}_j$ converges weakly in $L^2(Y)$ to $h\mathbf{g}$.*

The next two lemmas are more involved. Let $\mathcal{A}^n = \mathcal{A} \times \ldots \times \mathcal{A}$ denote the product of n copies of the disk algebra. The dual of $\mathcal{A}^n$ can then be identified with $\mathcal{C}^n$, the space of n-tuples of Cauchy transforms $\mathbf{f} = (f^1, \ldots, f^n)$, normed by $\|\mathbf{f}\|_{\mathcal{C}^n} = \sum_{i=1}^n \|f^i\|_{\mathcal{C}}$. The main technical argument needed for the proof of Theorem 3.5 is contained in the following lemma.

Lemma 3.3. *Let $\{\mathbf{f}_m\}_{m=1}^\infty$ be a sequence in $\mathcal{C}^n$ which converges weak-star to $\mathbf{f}$. There exists a subsequence $\{\mathbf{f}_{m_k}\}_{k=1}^\infty$ and a sequence of outer functions $M_k : \mathbb{D} \to \mathbb{C}$ satisfying the following properties:*

(i) $\|M_k\|_\infty \leq 1$,

(ii) M_k *converges uniformly on compacts to a non-zero outer function* M,

(iii) $M_k \mathbf{f}_{m_k} = (M_k f^1_{m_k}, M_k f^2_{m_k}, \ldots, M_k f^n_{m_k}) \in (H^2)^n$,

(iv) the sequence $\{M_k \mathbf{f}_{\mathbf{m_k}}\}_{m=1}^\infty$ *converges weakly to* $M\mathbf{f}$ *in* $(H^2)^n$.

Proof. Let $\mathbf{g} = (g^1, g^2, \ldots, g^n)$ be an n-tuple of Cauchy transforms and for some fixed choice of $p \in (1/2, 1)$ let

$$s(\zeta) = \max\Big(\sum_{i=1}^n |g^i(\zeta)|^p, 1\Big), \quad \zeta \in \mathbb{T}.$$

Then s is integrable on the circle and $\int_\mathbb{T} s\, dm \leq C_1 \|\mathbf{g}\|^p_{\mathcal{C}^n}$, where the constant $C_1 > 0$ depends on p and n, but is independent of $\mathbf{g} \in \mathcal{C}^n$. We let H be the Herglotz transform of s, that is

$$H(z) = \int_\mathbb{T} \frac{\zeta + z}{\zeta - z} s(\zeta) dm(\zeta).$$

Note that the real part of H is the Poisson extension of $s(\zeta)$ to $\mathbb{D}$. This shows that H has positive real part (hence is outer), $|H| \geq s \geq 1$ on $\mathbb{T}$ and $|H(z)| \geq 1$ for all $z \in \mathbb{D}$. We also have that $H(0) = \int_\mathbb{T} s\, dm \leq C_1 \|\mathbf{g}\|^p_{\mathcal{C}^n}$. Let $q = 2 - 2p \in (0, 1)$. For each $i \in \{1, \ldots, n\}$ we have the estimate

$$\int_\mathbb{T} |g^i/H|^2 dm \leq \int_\mathbb{T} |g^i/s|^2 dm \leq \int_\mathbb{T} |g^i|^{2-2p} dm = \|g^i\|^q_q \leq C_2 \|g^i\|^q_\mathcal{C}.$$

Since the functions g^i are in $\mathcal{C} \subset N^+$, the Smirnov maximum principle implies that $g^i/H \in H^2$, or equivalently $\mathbf{g}/H \in (H^2)^n$. Moreover, $\|\mathbf{g}/H\|_{(H^2)^n} \leq C \|\mathbf{g}\|^q_{\mathcal{C}^n}$, with constant $C > 0$ depending only on the fixed choice of p and the dimension n.

Let now $\{\mathbf{f}_m\}_{m=1}^\infty$ be a sequence in $\mathcal{C}^n$ which converges weak-star to $\mathbf{f}$, meaning that $\mathbf{f}_m$ converges pointwise to $\mathbf{f}$ in $\mathbb{D}$, and we have $\sup_m \|\mathbf{f}_m\|_{\mathcal{C}^n} < \infty$. For each integer $m \geq 1$ we construct the function $H = H_m$ as above. By what we have established above, $\{\mathbf{f}_m/H_m\}_{m=1}^{\infty}$ is a bounded sequence in $(H^2)^n$. Since $\|1/H_m\|_{\infty} \leq 1$ and $H_m(0) \leq C_1\|\mathbf{f}_m\|_{\mathcal{C}^n}^p$, there exists a subsequence $\{m_k\}_{k=1}^{\infty}$ such that $M_k = 1/H_{m_k}$ converges uniformly on compacts to a non-zero analytic function M. Then M has positive real part, since each of the functions M_k has positive real part, and therefore M is outer. The sequence $\{M_k\mathbf{f}_{m_k}\}_{k=1}^{\infty}$ is bounded in $(H^2)^n$ and converges pointwise to the function $M\mathbf{f}$, which is equivalent to weak convergence in $(H^2)^n$. □

For the rest of the section, let $\mathcal{H} = \mathcal{H}[\mathbf{B}]$ be a fixed space of X-valued functions, where X is a finite dimensional Hilbert space. Thus $\mathbf{B}$ takes values in $\mathcal{B}(Y, X)$ for some auxilliary Hilbert space Y. As before, set $\Delta(\zeta) = (I_Y - \mathbf{B}(\zeta)^*\mathbf{B}(\zeta))^{1/2}$ for $\zeta \in \mathbb{T}$. Fix an orthonormal basis $\{e_i\}_{i=1}^n$ for the finite dimensional Hilbert space X. We can define a map from $H^2(X)$ to $(H^2)^n$ by the formula $\mathbf{f} \mapsto (f^i)_{i=1}^n$ where the components f^i are the coordinate functions $f^i(z) = \langle \mathbf{f}(z), e_i \rangle_X$. Then a function $\mathbf{f} \in H^2(X)$ has a continuous extension to clos($\mathbb{D}$) if and only if all of its coordinate functions f^i are contained in the disk algebra $\mathcal{A}$.

Lemma 3.4. *Let $S \subset \mathcal{C}^n \oplus$ clos$(\Delta H^2(Y))$ be the linear manifold consisting of tuples of the form*

$$(\mathbf{B}\mathbf{f}, \Delta\mathbf{f})$$

for some $\mathbf{f} \in N^+(Y)$. Then S is weak-star closed in $\mathcal{C}^n \oplus$ clos$(\Delta H^2(Y))$.

Proof. Since $\mathcal{A}^n \oplus$ clos$(\Delta H^2(Y))$ is separable, Krein-Smulian theorem implies that it is enough to check weak-star sequential closedness of the set S. Thus, let the sequence

$$\{(\mathbf{B}\mathbf{f}_m, \Delta\mathbf{f}_m)\}_{m=1}^{\infty} = \{(\mathbf{h}_m, \mathbf{g}_m)\}_{m=1}^{\infty} \subset \mathcal{C}^n \oplus \text{clos}(\Delta H^2(Y))$$

converge in the weak-star topology to $(\mathbf{h}, \mathbf{g})$. Then $\{\mathbf{g}_m\}_{m=1}^{\infty}$ converges weakly in the Hilbert space clos$(\Delta H^2(Y))$, and by passing to a subsequence and next to the Cesàro means of that subsequence, we can assume that the sequence $\{\mathbf{g}_m\}_{m=1}^{\infty}$ converges to $\mathbf{g}$ in the norm. By applying Lemma 3.3 and Lemma 3.2 we obtain a sequence of outer functions $\{M_k\}_{k=1}^{\infty}$ and an outer function M such that $\{(M_k\mathbf{h}_{m_k}, M_k\mathbf{g}_{m_k})\}_{k=1}^{\infty}$ converges weakly in the Hilbert space $H^2(X) \oplus$ clos$(\Delta H^2(Y))$ to $(M\mathbf{h}, M\mathbf{g})$. Note that

$$(M_k\mathbf{h}_{m_k}, M_k\mathbf{g}_{m_k}) = (\mathbf{B}M_k\mathbf{f}_{m_k}, \Delta M_k\mathbf{f}_{m_k}),$$

and

$$\int_{\mathbb{T}} \|M_k\mathbf{f}_{m_k}\|_Y^2 dm = \int_{\mathbb{T}} \|\mathbf{B}M_k\mathbf{f}_{m_k}\|_X^2 dm + \int_{\mathbb{T}} \|\Delta M_k\mathbf{f}_{m_k}\|_Y^2 dm < \infty.$$

Since $M_k\mathbf{f}_{m_k}$ is in $N^+(Y)$, the Smirnov maximum principle implies that we have $M_k\mathbf{f}_{m_k} \in H^2(Y)$, and consequently the tuples $(\mathbf{B}M_k\mathbf{f}_{m_k}, \Delta M_k\mathbf{f}_{m_k})$ are contained in the closed subspace $U = \{(\mathbf{B}\mathbf{h}, \Delta\mathbf{h}) : \mathbf{h} \in H^2(Y)\}$. It follows that the weak limit $(M\mathbf{h}, M\mathbf{g})$ is also contained in U, and hence $(M\mathbf{h}, M\mathbf{g}) = (\mathbf{Bf}, \Delta\mathbf{f})$ for some $\mathbf{f} \in H^2(Y)$. Then

$$(\mathbf{h}, \mathbf{g}) = \Big(\mathbf{B}\frac{\mathbf{f}}{M}, \Delta\frac{\mathbf{f}}{M}\Big),$$

where $\mathbf{f}/M \in N^+(Y)$. □

Theorem 3.5. *Assume that $\mathcal{H}[\mathbf{B}]$ consists of functions taking values in a finite dimensional Hilbert space. Then the set of functions in $\mathcal{H}[\mathbf{B}]$ which extend continuously to clos$(\mathbb{D})$ is dense in the space.*

Proof. Let $K = H^2(X) \oplus \text{clos}(\Delta H^2(Y))$. Recall from Theorem 2.2 that the space $\mathcal{H}[\mathbf{B}]$ is equipped with an isometric embedding J where the tuple $J\mathbf{f} = (\mathbf{f}, \mathbf{g}) \in K$ is uniquely determined by the requirement for it be orthogonal to

$$U = \{(\mathbf{Bh}, \Delta\mathbf{h} : \mathbf{h} \in H^2(Y)\} \subset K.$$

We identify functions $\mathbf{f} \in \mathcal{H}[\mathbf{B}]$ with their coordinates $(f^1, \dots, f^n)$ with respect to the fixed orthonormal basis of X. Now assume that $\mathbf{f} \in \mathcal{H}[\mathbf{B}]$ is orthogonal to any function in $\mathcal{H}[\mathbf{B}]$ which extends continuously to clos$(\mathbb{D})$, i.e., that $\mathbf{f}$ is orthogonal to $\mathcal{H}[\mathbf{B}] \cap \mathcal{A}^n$. We shall show that $J\mathbf{f} = (\mathbf{Bh}, \Delta\mathbf{h})$ for some $\mathbf{h} \in H^2(Y)$, which implies that $\mathbf{f} = 0$. Consider $J(\mathcal{A}^n \cap \mathcal{H}[\mathbf{B}])$ as a subspace of $\mathcal{A}^n \oplus \text{clos}(\Delta H^2(Y))$. For each $\mathbf{h} \in H^2(Y)$, let

$$l_{\mathbf{h}} = (\mathbf{Bh}, \Delta\mathbf{h}) \in \mathcal{C}^n \oplus \text{clos}(\Delta H^2(Y))$$

be a functional on $\mathcal{A}^n \oplus \text{clos}(\Delta H^2(Y))$, acting as usual by integration on the boundary $\mathbb{T}$. We claim that

$$J(\mathcal{A}^n \cap \mathcal{H}[\mathbf{B}]) = \cap_{\mathbf{h} \in H^2(Y)} \ker l_{\mathbf{h}}.$$

Indeed, if $\mathbf{f} \in \mathcal{A}^n \cap \mathcal{H}[\mathbf{B}]$, then for any functional $l_{\mathbf{h}}$ we have

$$l_{\mathbf{h}}(J\mathbf{f}) = \big\langle J\mathbf{f}, (\mathbf{Bh}, \Delta\mathbf{h})\big\rangle = 0,$$

because $J\mathbf{f}$ is orthogonal to U. Conversely, if the tuple $(\mathbf{f}, \mathbf{g}) \in \mathcal{A}^n \oplus \text{clos}(\Delta H^2(Y))$ is contained in $\cap_{\mathbf{h} \in H^2(Y)} \ker l_{\mathbf{h}}$, then $(\mathbf{f}, \mathbf{g}) \in K$ is orthogonal to U, and hence $\mathbf{f} \in \mathcal{A}^n \cap \mathcal{H}[\mathbf{B}]$ by Theorem 2.2. Now, viewed as an element of $\mathcal{C}^n \oplus \text{clos}(\Delta H^2(Y))$, the tuple $J\mathbf{f}$ annihilates $J(\mathcal{A}^n \cap \mathcal{H}[\mathbf{B}])$, and so by Lemma 3.1 lies in the weak-star closure of linear manifold of functionals of the form $l_{\mathbf{h}}$. Thus Lemma 3.4 implies that $J\mathbf{f} = (\mathbf{Bh}, \Delta\mathbf{h})$ for some $\mathbf{h} \in N^+(Y)$. The Smirnov maximum principle and the computation

$$\int_{\mathbb{T}} \|\mathbf{h}\|_Y^2 dm(\zeta) = \int_{\mathbb{T}} \|\mathbf{Bh}\|_X^2 dm + \int_{\mathbb{T}} \|\Delta\mathbf{h}\|_Y^2 dm < \infty$$

show that $\mathbf{h} \in H^2(Y)$. Hence $J\mathbf{f} \in (JH(\mathbf{B})^{\perp}$, so that $\mathbf{f} = 0$ and the proof is complete. □

Theorem 1.1 of the introduction now follows as a consequence of the next result, which is an easy extension of Theorem 3.5.

Corollary 3.6. *Assume that $\mathcal{H}[\mathbf{B}]$ consists of functions taking values in a finite dimensional Hilbert space. If M is any L-invariant subspace of $\mathcal{H}[\mathbf{B}]$, then the set of functions in M which extend continuously to clos$(\mathbb{D})$ is dense in M.*

Proof. If M contains the constant vectors, then Proposition 2.1 applies, and hence M is of the type $\mathcal{H}(\mathbf{B}_0)$ for some contractive function $\mathbf{B}_0$. Then the result follows immediately from Theorem 3.5. If constant vectors are not contained in M, then let

$$M^+ = \{\mathbf{f} + c : \mathbf{f} \in M, c \in X\}.$$

The subspace M^+ is closed, as it is a sum of a closed subspace and a finite dimensional space. Moreover, closed graph theorem implies that the skewed projection $P : M^+ \to M$ taking $\mathbf{f} + c$ to $\mathbf{f}$ is bounded. The theorem holds for M^+, so if $\mathbf{f} \in M$, then there exists constants c_n and functions $\mathbf{f}_n \in M$ such that $\mathbf{h}_n = \mathbf{f}_n + c_n$ is continuous on clos$(\mathbb{D})$, and $\mathbf{h}_n$ tends to $\mathbf{f}$ in the norm of $\mathcal{H}$. Consequently, the functions $\mathbf{f}_n = \mathbf{h}_n - c_n$ are continuous on clos$(\mathbb{D})$, and we have that $\mathbf{f}_n = P\mathbf{h}_n$ tends to $P\mathbf{f} = \mathbf{f}$ in the norm of $\mathcal{H}$. □

4 Applications of the density theorem

We temporarily leave the the main subject in order to present applications of Theorem 3.5 and Corollary 3.6. All Hilbert spaces of analytic functions will be assumed to satisfy (A.1)-(A.3).

4.1 M_z-invariant subspaces.

Corollary 1.2 stated in the introduction is now an easy consequence of Theorem 3.5. We restate the theorem for the reader's convenience.

Corollary 4.1. *Let $\mathcal{H}$ be a Hilbert space of analytic functions which satisfies (A.1)-(A.3) and is invariant for the forward shift M_z. For a closed M_z-invariant subspace $\mathcal{M}$ of $\mathcal{H}$ with $\dim \mathcal{M} \ominus M_z\mathcal{M} = n < \infty$, let $\varphi_1, \dots \varphi_n$ be an orthonormal basis in $\mathcal{M} \ominus M_z\mathcal{M}$, and denote by ϕ the corresponding row operator-valued function. Then*

$$\mathcal{M} = \phi\mathcal{H}[\mathbf{C}], \tag{15}$$

where $\mathcal{H}[\mathbf{C}]$ consists of $\mathbb{C}^n$-valued functions and the mapping $g \mapsto \phi g$ is an isometry from $\mathcal{H}[\mathbf{C}]$ onto $\mathcal{M}$. Moreover,

$$\{\sum_{i=0}^{n} \varphi_i u_i : \; u_i \in \mathcal{A}, \; 1 \leq i \leq n\} \cap \mathcal{H} \tag{16}$$

is a dense subset of $\mathcal{M}$.

Proof. For any $f \in \mathcal{M}$ we have that $f(z) - \sum_{i=1}^{n} \langle f, \phi_i \rangle_{\mathcal{H}} \phi_i(z) \in M_z\mathcal{M}$. Thus the operator $L^\phi : \mathcal{M} \to \mathcal{M}$ given by

$$L^\phi f(z) = \frac{f(z) - \sum_{i=1}^{n} \langle f, \phi_i \rangle_{\mathcal{H}} \phi_i(z)}{z}$$

is well-defined, and it is a contraction since it is a composition of a projection with the contractive operator L. A straightforward computation shows that for $\lambda \in \mathbb{D}$ the following equation holds:

$$(1 - \lambda L^\phi)^{-1} f(z) = \frac{z f(z) - \lambda \sum_{i=1}^{n} \langle (1 - \lambda L^\phi)^{-1} f, \phi_i \rangle_{\mathcal{H}} \phi_i(z)}{z - \lambda}.$$

Thus the analytic function in the numerator on the right-hand side above must have a zero at $z = \lambda$. It follows that $f(\lambda) = \sum_{i=1}^{n} \langle (1 - \lambda L^\phi)^{-1} f, \phi_i \rangle_{\mathcal{H}} \phi_i(\lambda)$. Consider now the mapping U taking $f \in \mathcal{M}$ to the vector $Uf(\lambda) = \Big(\langle (1 - \lambda L^\phi)^{-1} f, \phi_i \rangle_{\mathcal{H}} \Big)_{i=1}^{n}$ and let $\mathcal{M}_0 = U\mathcal{M}$ with the norm on $\mathcal{M}_0$ which makes $U : \mathcal{M} \to \mathcal{M}_0$ a unitary mapping. Then $\mathcal{M}_0$ is a space of $\mathbb{C}^n$-valued analytic functions which satisfies (A.1')-(A.3') and to which Theorem 3.5 applies. The claims in the statement follow immediately from this. □

4.2 Reverse Carleson measures. A finite Borel measure on clos($\mathbb{D}$) is a *reverse Carleson measure* for $\mathcal{H}$ if there exists a constant $C > 0$ such that the estimate

$$\|f\|_{\mathcal{H}}^2 \leq C \int_{\text{clos}(\mathbb{D})} |f(z)|^2 d\mu(z) \tag{17}$$

holds for f which belong to some dense subset of $\mathcal{H}$ and for which the integral on the right-hand side makes sense, e.g. by the existence of radial boundary values of f on the support of the singular part of μ on $\mathbb{T}$. For the class of spaces considered in this paper it is natural to require, due to Theorem 3.5, that (17) holds for all functions in $\mathcal{H}$ which admit continuous extensions to clos($\mathbb{D}$).

Our main result in this context characterizes the existence of a reverse Carleson measures for spaces which are invariant for M_z. If such a measure exists, then we can moreover identify one which is in a sense minimal.

Theorem 4.2. *Let $\mathcal{H} = \mathcal{H}[\mathbf{B}]$ be invariant for M_z. Then the following are equivalent.*

(i) $\mathcal{H}$ admits a reverse Carleson measure.

(ii)

$$\sup_{0<r<1} \int_{\mathbb{T}} \left\| \frac{\sqrt{1 - |r\lambda|^2}}{1 - r\overline{\lambda} z} \right\|_{\mathcal{H}}^2 dm(\lambda) < \infty.$$

(iii) If k is the reproducing kernel of $\mathcal{H}$, then

$$\sup_{0<r<1} \int_{\mathbb{T}} \frac{1}{(1-|r\lambda|^2)k(r\lambda, r\lambda)} dm(\lambda) < \infty.$$

If the above conditions are satisfied, then

$$h_1(\lambda) := \lim_{r\to 1} \left\| \frac{\sqrt{1-|r\lambda|^2}}{1-r\bar{\lambda}z} \right\|_{\mathcal{H}}^2$$

and

$$h_2(\lambda) := \lim_{r\to 1} \frac{1}{(1-|r\lambda|^2)k(r\lambda, r\lambda)}$$

define reverse Carleson measures for $\mathcal{H}$. Moreover, if ν is any reverse Carleson measure for $\mathcal{H}$ and v is the density of the absolutely continuous part of the restriction of ν to $\mathbb{T}$, then $h_1 dm$ and $h_2 dm$ have the following minimality property: there exist constants $C_i > 0, i = 1, 2$ such that

$$h_i(\lambda) \le C_i v(\lambda)$$

for almost every $\lambda \in \mathbb{T}$.

Proof. (i) $\Rightarrow$ (ii): Let ν be a reverse Carleson measure for $\mathcal{H}$. First, we show that we can assume that ν is supported on $\mathbb{T}$. For this, we will use the inequality

$$\|z^n f\|_{\mathcal{H}}^2 \le C \int_{\mathbb{D}} |z^n f(z)|^2 d\nu(z) + C \int_{\mathbb{T}} |f(z)|^2 d\nu(z).$$

Since $Lzf = f$ and L is a contraction, it follows that $\|zf\|_{\mathcal{H}} \ge \|f\|_{\mathcal{H}}$, and thus letting n tend to infinity in the above inequality we obtain

$$\|f\|_{\mathcal{H}}^2 \le C \int_{\mathbb{T}} |f(z)|^2 d(\nu|\mathbb{T})(z).$$

Thus we might replace ν by $\nu|\mathbb{T}$, as claimed. Next, we note that $H^\infty \subset \mathcal{H}$. Indeed, $\mathcal{H}$ contains 1 and is M_z-invariant, thus contains the polynomials. If p_n is a uniformly bounded sequence of polynomials converging pointwise to $f \in H^\infty$, then the existence of a reverse Carleson measure ensures that the norms $\|p_n\|_{\mathcal{H}}$ are uniformly bounded, and thus a subsequence of $\{p_n\}$ converges weakly to $f \in \mathcal{H}$. Thus, the function $z \mapsto \frac{1}{1-\bar{\lambda}z}$ is contained in $\mathcal{H}$ for each $\lambda \in \mathbb{D}$. Define

$$H(\lambda) =: \int_{\mathbb{T}} \frac{1-|\lambda|^2}{|1-\bar{\lambda}z|^2} d\nu(z), \quad \lambda \in \mathbb{D},$$

which is positive and harmonic in $\mathbb{D}$, and since ν is a reverse Carleson measure for $\mathcal{H}$, there exists a constant $C > 0$ such that

$$\left\| \frac{\sqrt{1-|\lambda|^2}}{1-\overline{\lambda}z} \right\|^2_{\mathcal{H}} \leq CH(\lambda). \tag{18}$$

The implication now follows from the mean value property of harmonic functions.

(ii) ⇒ (iii): There exists an orthogonal decomposition

$$\frac{1}{1-\overline{\lambda}z} = \frac{1}{1-|\lambda|^2}\frac{k(\lambda,z)}{k(\lambda,\lambda)} + g(z),$$

where g is some function which vanishes at λ. Thus

$$\left\|\frac{1}{1-\overline{\lambda}z}\right\|^2_{\mathcal{H}} = \frac{1}{(1-|\lambda|^2)^2 k(\lambda,\lambda)} + \|g\|^2$$

and consequently

$$\frac{1}{(1-|\lambda|^2)k(\lambda,\lambda)} \leq \left\| \frac{\sqrt{1-|\lambda|^2}}{1-\overline{\lambda}z} \right\|^2_{\mathcal{H}}. \tag{19}$$

Thus (iii) follows from (ii).

(iii) ⇒ (i): If $\mathcal{H} = \mathcal{H}[\mathbf{B}]$ with $\mathbf{B} = (b_i)_{i=1}^{\infty}$, then let w be the outer function with boundary values satisfying $|w(\zeta)|^2 = 1-\sum_{i\in I}|b_i(\zeta)|^2$. The existence of such a function is ensured by (iii). The space $M = wH^2 = \{f = wg : g \in H^2\}$ normed by $\|f\|_M = \|g\|_2$ is a Hilbert space of analytic functions with a reproducing kernel given by

$$K_M(\lambda,z) = \frac{\overline{w(\lambda)}w(z)}{1-\overline{\lambda}z}.$$

It is not hard to verify that $K = k_{\mathcal{H}} - K_M$ is a positive-definite kernel. Then $k_{\mathcal{H}} = K + K_M$, and hence M is contained contractively in $\mathcal{H}$. Thus for any function $f \in M$ we have that $\|f\|^2_{\mathcal{H}} \leq \|f\|^2_M = \int_{\mathbb{T}} \frac{|f|^2}{|w|^2}dm$, and $|w|^{-2}dm$ will be a reverse Carleson measure if M is dense in $\mathcal{H}$. But $1/w \in H^2$ by (iii), and thus $\mathcal{H}\cap\mathcal{A} \subset M$, so M is indeed dense in $\mathcal{H}$.

The limits defining h_1 and h_2 exist as a consequence of general theory of boundary behaviour of subharmonic functions. The inequality $h_1(\lambda) \leq C_1 v(\lambda)$ is seen immediately from (18) and $h_1(\lambda) \leq C_2 v(\lambda)$ is then seen from (19). □

We remark that if $\mathcal{H}$ is not M_z-invariant, then the space might admit a reverse Carleson measure even though (iii) is violated. An example is any L-invariant proper subspace of H^2.

An application of part (iii) of Theorem 4.2 to $\mathcal{H} = \mathcal{H}(b)$, with b non-extreme, lets us deduce a result essentially contained in [10], namely that $\mathcal{H}(b)$ admits a reverse Carleson measure if and only if $(1-|b|^2)^{-1} \in L^1(\mathbb{T})$, and the measure $\mu = (1-|b|^2)^{-1}dm$ is then a minimal reverse Carleson measure in the sense made precise by the theorem. A second application is to Dirichlet-type spaces. Recall from Section 2.5 that for μ a positive finite Borel measure supported on $\mathrm{clos}(\mathbb{D})$, the Hilbert space $\mathcal{D}(\mu)$ is defined as the completion of the analytic polynomials under the norm

$$\|f\|^2_{\mathcal{D}(\mu)} = \|f\|_2^2 + \int_{\mathbb{T}}\int_{\mathrm{clos}(\mathbb{D})} \frac{|f(z)-f(\lambda)|^2}{|z-\lambda|^2} d\mu(z)dm(\lambda).$$

Corollary 4.3. *The space $D(\mu)$ admits a reverse Carleson measure if and only if*

$$\int_{clos(\mathbb{D})} \frac{d\mu(z)}{1-|z|^2} < \infty.$$

The minimal reverse Carleson measure (in the sense of Theorem 4.2) is given by $d\nu = hdm$ where

$$h(\lambda) = 1 + \int_{clos(\mathbb{D})} \frac{d\mu(z)}{|1-\overline{\lambda}z|^2}, \lambda \in \mathbb{T}.$$

Proof. The space $\mathcal{D}(\mu)$ satisfies the assumptions of Theorem 4.2. A computation shows that if $k_\lambda(z) = \frac{1}{1-\overline{\lambda}z}$, then

$$(1-|\lambda|^2)\|k_\lambda\|^2_{\mathcal{D}(\mu)} = 1 + \int_{\mathrm{clos}(\mathbb{D})} \frac{|\lambda|^2}{|1-\overline{\lambda}z|^2} d\mu(z).$$

The claim now follows easily from (ii) of Theorem 4.2 and Fubini's theorem. □

It is interesting to note that the condition on μ above holds even in cases when $\mathcal{D}(\mu)$ is strictly contained in H^2 (see [3]).

Our last result in the context of reverse Carleson measures is a non-existence result which answers in particular a question posed in [10].

Theorem 4.4. *Assume that the identity*

$$\|Lf\|^2_{\mathcal{H}} = \|f\|^2_{\mathcal{H}} - |f(0)|^2$$

holds in $\mathcal{H}$. If $\mathcal{H}$ admits a reverse Carleson measure, then $\mathcal{H}$ is isometrically contained in the Hardy space H^2.

Proof. We will use Theorem 2.2 and Theorem 2.6. The limit in part (ii) of Theorem 2.6 vanishes, because the norm identity implies that $\|zL_\lambda f\| = \|L_\lambda f\|_{\mathcal{H}}$. It will thus suffice to show that the limit in part (iii) of said theorem also vanishes, namely that

$$\lim_{r\to 1} \int_{\mathbb{T}} (1-r^2)\|L_{r\lambda}f\|^2_{\mathcal{H}} dm(\lambda) = 0. \tag{20}$$

If $f \in \mathcal{H}$ is continuous in $\operatorname{clos}(\mathbb{D})$ then so is $L_{r\lambda} f$, and if μ is a reverse Carleson measure for $\mathcal{H}$, then we have the estimate

$$\begin{aligned} &\lim_{r \to 1} \int_{\mathbb{T}} (1 - r^2) \|L_{r\lambda} f\|^2_{\mathcal{H}} dm(\lambda) \\ &\leq C \limsup_{r \to 1} \int_{\operatorname{clos}(\mathbb{D})} \int_{\mathbb{T}} \frac{1 - r^2}{|\zeta - r\lambda|^2} |f(\zeta) - f(r\lambda)|^2 dm(\lambda) d\mu(\zeta) \end{aligned} \tag{21}$$

Since f is continuous and bounded in $\operatorname{clos}(\mathbb{D})$, we readily deduce from standard properties of the Poisson kernel and bounded analytic functions that the functions

$$G_r(\zeta) = \int_{\mathbb{T}} \frac{1 - r^2}{|\zeta - r\lambda|^2} |f(\zeta) - f(r\lambda)|^2 dm(\lambda)$$

are uniformly bounded in $\operatorname{clos}(\mathbb{D})$ and that $\lim_{r \to 1} G_r(\zeta) = 0$ for all $\zeta \in \operatorname{clos}(\mathbb{D})$. Dominated convergence theorem now implies that the limit in (21) is 0, and so (20) holds. □

From part (ii) of Theorem 2.2 we easily deduce that the condition on L of Theorem 4.4 is equivalent to $\Theta H^2(Y_1) = \{0\}$ in (8). In the case that $\mathbf{B} = b$ is a scalar-valued function, this occurs if and only if b is an extreme point of the unit ball of H^∞. Thus an extreme point b which is not an inner function cannot generate a space $\mathcal{H}(b)$ which admits a reverse Carleson measure. We remark also that Theorem 4.4 holds, with the same proof, even when $\mathcal{H}$ consists of functions taking values in a finite dimensional Hilbert space X, where the norm identity then instead reads $\|L\mathbf{f}\|^2 = \|\mathbf{f}\|^2 - \|\mathbf{f}(0)\|^2_X$, and definition of reverse Carleson measure is extended naturally to the vector-valued setting.

4.3 Formula for the norm in a nearly invariant subspace. A space $\mathcal{M}$ is *nearly invariant* if whenever $\lambda \in \mathbb{D}$ is not a common zero of the functions in $\mathcal{M}$ and $f(\lambda) = 0$ for some $f \in \mathcal{M}$, then $\frac{f(z)}{z - \lambda} \in \mathcal{M}$. If $\mathcal{H}$ is M_z-invariant, then an example of a nearly invariant subspace is any M_z-invariant subspace for which $\dim \mathcal{M} \ominus M_z\mathcal{M} = 1$. The concept of a nearly invariant subspace has appeared in [23], and have since been used as a tool in solutions to numerous problems in operator theory.

We will now prove a formula for the norm of functions contained in a nearly invariant subspace $\mathcal{M}$ which is similar to Proposition 2.5 but which is better suited for exploring the structure of $\mathcal{M}$.

Proposition 4.5. *Let $\mathcal{M} \subseteq \mathcal{H}$ be a nearly invariant subspace and k be the common order of the zero at 0 of the functions in $\mathcal{M}$. Let $\phi \in \mathcal{M}$ be the function satisfying $\langle f, \phi \rangle_{\mathcal{H}} = \frac{f^{(k)}(0)}{\phi^{(k)}(0)}$ for all $f \in \mathcal{M}$, $L^\phi : \mathcal{M} \to \mathcal{M}$ be the contractive operator given by*

$$L^\phi f(z) = \frac{f(z) - \langle f, \phi \rangle_{\mathcal{H}} \phi(z)}{z},$$

and $L^{\phi}_{\lambda} = L^{\phi}(1-\lambda L^{\phi})^{-1}$, $\lambda \in \mathbb{D}$. *If the sequence* $\{L^n\}_{n=1}^{\infty}$ *converges to zero in the strong operator topology on* $\mathcal{H}$, *then*

$$\|f\|_{\mathcal{H}}^2 = \|f/\phi\|_2^2 + \lim_{r\to 1}\int_{\mathbb{T}} \|zL^{\phi}_{r\lambda}f\|_{\mathcal{H}}^2 - \|L^{\phi}_{r\lambda}f\|_{\mathcal{H}}^2 dm(\lambda). \tag{22}$$

Proof. The fact that L^{ϕ} maps $\mathcal{M}$ into itself follows easily by nearly invariance of $\mathcal{M}$. The formula

$$\|f\|_{\mathcal{H}}^2 = \|f/\phi\|_2^2 + \lim_{r\to 1}\int_{\mathbb{T}} \|zL^{\phi}_{r\lambda}f\|_{\mathcal{H}}^2 - r^2\|L^{\phi}_{r\lambda}f\|_{\mathcal{H}}^2 dm(\lambda) \tag{23}$$

holds in much more general context and without the assumption on convergence of $\{L^n\}_{n=1}^{\infty}$ to zero. For its derivation we refer to Lemma 2.2 of [7], and the discussion succeeding it. The equation (22) will follow from (23) if we can show that the additional assumption on L implies that

$$\lim_{r\to 1}\int_{\mathbb{T}} (1-r^2)\|L^{\phi}_{r\lambda}f\|_{\mathcal{H}}^2 dm(\lambda) = 0.$$

To this end, observe that for $\lambda \in \mathbb{D}$ we have

$$L^{\phi}_{\lambda}f(z) = \frac{f(z)-f(\lambda)}{z-\lambda} - \frac{f(\lambda)}{\phi(\lambda)}\frac{\phi(z)-\phi(\lambda)}{z-\lambda} = L_{\lambda}f(z) - \frac{f(\lambda)}{\phi(\lambda)}L_{\lambda}\phi(z),$$

and so if $f/\phi \in H^{\infty}$, then the argument in the proof of part (iii) of Theorem 2.6 shows that

$$\lim_{r\to 1^-}\int_{\mathbb{T}} (1-r^2)\|L^{\phi}_{r\lambda}f\|_{\mathcal{H}}^2 dm(\lambda)$$

$$\lesssim \lim_{r\to 1^-}\int_{\mathbb{T}} (1-r^2)\|L_{r\lambda}f\|_{\mathcal{H}}^2 dm(\lambda) + \|f/\phi\|_{\infty}^2\int_{\mathbb{T}} (1-r^2)\|L_{r\lambda}\phi\|_{\mathcal{H}}^2 dm(\lambda) = 0.$$

Next, construct the Hilbert space $\tilde{\mathcal{M}} = \mathcal{M}/\phi = \{f/\phi : f \in \mathcal{M}\}$ with the norm $\|f/\phi\|_{\tilde{\mathcal{M}}} = \|f\|_{\mathcal{H}}$. It is easy to see that $\tilde{\mathcal{M}}$ satisfies (A.1)-(A.3), and thus by Theorem 3.5 the functions with continuous extensions to clos($\mathbb{D}$) form a dense subset of $\tilde{\mathcal{M}}$. Since multiplication by ϕ is a unitary map from $\tilde{\mathcal{M}}$ to $\mathcal{M}$, we see that functions $f \in \mathcal{M}$ such that f/ϕ has continuous extension to clos($\mathbb{D}$) form a dense subset of $\mathcal{M}$. Finally, consider the mapping

$$\mathbf{Q}f = \limsup_{r\to 1^-}\int_{\mathbb{T}} (1-r^2)\|L^{\phi}_{r\lambda}f\|_{\mathcal{H}}^2 dm(\lambda), \quad f \in \mathcal{M}.$$

Then $\mathbf{Q}(f+g) \leq 2(\mathbf{Q}f + \mathbf{Q}g)$, and we have shown above that $\mathbf{Q} \equiv 0$ on a dense subset of $\mathcal{M}$. A peek at (23) reveals that $\mathbf{Q}f \leq \|f\|_{\mathcal{H}}^2$, and so $\mathbf{Q}$ is continuous on $\mathcal{M}$. Thus $\mathbf{Q} \equiv 0$ since it vanishes on a dense subset. $\square$

We will see an application of Proposition 4.5 in the sequel. For now, we show how it can be used to deduce a theorem of Hitt on the structure of nearly invariant subspaces of H^2 (see [23]).

Corollary 4.6. *If a closed subspace $\mathcal{M} \subset H^2$ is nearly invariant, then it is of the form $\mathcal{M} = \phi K_\theta$, where $K_\theta = H^2 \ominus \theta H^2$ is an L-invariant subspace of H^2, and we have the norm equality $\|f/\phi\|_2 = \|f\|_2, f \in \mathcal{M}$.*

Proof. If $\mathcal{M}$ is nearly invariant then the formula (22) gives $\|f/\phi\|_2 = \|f\|_2$, since M_z is an isometry on H^2. Thus $\mathcal{M}/\phi := \{f/\phi : f \in \mathcal{M}\}$ is closed in H^2, and it is easy to see that it is L-invariant. Thus $\mathcal{M}/\phi = K_\theta$ by Beurling's famous characterization. □

4.4 Orthocomplements of shift invariant subspaces of the Bergman space.

The Bergman space $L^2_a(\mathbb{D})$ consists of functions $g(z) = \sum_{k=0} g_k z^k$ analytic in $\mathbb{D}$ which satisfy

$$\|g\|^2_{L^2_a(\mathbb{D})} = \int_{\mathbb{D}} |g(z)|^2 dA(z) = \sum_{k=0}^{\infty} (k+1)^{-1} |g_k|^2 < \infty,$$

where dA denotes the normalized area measure on $\mathbb{D}$. The Bergman space is invariant for M_z and the lattice of M_z-invariant subspaces of $L^2_a(\mathbb{D})$ is well-known to be very complicated (see for example Chapter 8 and 9 of [15] and Chapters 6, 7 and 8 of [21]). Before stating and proving our next result, we will motivate it by showing that the orthogonal complements of M_z-invariant subspaces of $L^2_a(\mathbb{D})$ can consist entirely of ill-behaved functions. The following result is essentially due to A. Borichev [11].

Proposition 4.7. *There exists a subspace $\mathcal{M} \subsetneq L^2_a(\mathbb{D})$ which is invariant for M_z, with the property that for any non-zero function $g \in \mathcal{M}^\perp$ and any $\delta > 0$ we have*

$$\int_{\mathbb{D}} |g|^{2+\delta} dA = \infty.$$

Proof. The argument is based on the existence of a function f such that $f, 1/f \in L^2_a(\mathbb{D})$, yet if $\mathcal{M}$ is the smallest M_z-invariant subspace containing f, then $\mathcal{M} \neq L^2_a(\mathbb{D})$. Such a function exists by [12]. Take $g \in \mathcal{M}^\perp$ and assume that $\int_{\mathbb{D}} |g|^{2+\delta} dA < \infty$ for some $\delta > 0$. Since $\mathcal{M}$ is M_z-invariant, for any polynomial p we have that

$$\int_{\mathbb{D}} p f \overline{g} dA = 0. \tag{24}$$

By the assumption on g and Hölder's inequality we have that $f\overline{g} \in L^r(\mathbb{D})$ for $r > 1$ sufficiently close to 1. Let $s = \frac{r}{r-1}$ be the Hölder conjugate index of r. Since $1/f \in L^2_a(\mathbb{D})$, the analytic function $f^{-\epsilon}$ is in the Bergman space $L^s_a(\mathbb{D})$ for sufficiently small

$\epsilon > 0$, and hence can be approximated in the norm of $L^s_a(\mathbb{D})$ by a sequence $\{p_n\}_{n=1}^{\infty}$ of polynomials. If p is any polynomial, then

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{D}} |(p_n - f^{-\epsilon})pfg|dA \\ \lesssim &\lim_{n\to\infty} \|p_n - f^{-\epsilon}\|_{L^s(\mathbb{D})}\|pfg\|_{L^r(\mathbb{D})} = 0.\end{aligned}$$

By choosing $\epsilon = 1/K$ for sufficiently large positive integer K, we see from the above that for any polynomial p we have

$$\int_{\mathbb{D}} pf^{1-1/K}\overline{g}dA = 0,$$

which is precisely (24) with $\overline{g}$ replaced by $f^{-1/K}\overline{g}$, and of course we still have $ff^{-1/K}\overline{g} \in L^r(\mathbb{D})$ for the same choice of $r > 1$. Repeating the argument gives

$$\int_{\mathbb{D}} pf^{1-2/K}\overline{g}dA = 0,$$

and after K repetitions of the argument we arrive at

$$\int_{\mathbb{D}} p\overline{g}dA(z) = 0$$

for any polynomial p. Then $g = 0$ by density of polynomials in $L^2_a(\mathbb{D})$. □

Despite the rather dramatic situation described by Proposition 4.7, an application of Corollary 3.6 yields the following result.

Corollary 4.8. *Let $\mathcal{M}$ be a subspace of $L^2_a(\mathbb{D})$ which is invariant for M_z. Then the functions in the orthocomplement $\mathcal{M}^\perp$ which are derivatives of functions in the disk algebra $\mathcal{A}$ are dense in $\mathcal{M}^\perp$.*

Proof. The operator U given by

$$Uf(z) = \frac{1}{z}\int_0^z f(w)\,dw$$

is a unitary map between $L^2_a(\mathbb{D})$ and the classical Dirichlet space $\mathcal{D}$, and Corollary 3.6 applies to the latter space. A computation involving the Taylor series coefficients shows that $UM_z^*U^* = L$, and so if $\mathcal{M} \subset L^2_a(\mathbb{D})$ is M_z-invariant, then $U\mathcal{M}^\perp \subset \mathcal{D}$ is L-invariant. Thus from Corollary 3.6 we infer that the set of functions $f \in \mathcal{M}^\perp$ for which $Uf(z)$ is in the disk algebra $\mathcal{A}$ is dense in $\mathcal{M}^\perp$, and for any function f in this set we have $f(z) = (zUf(z))'$, so f is the derivative of a function in $\mathcal{A}$. □

5 Finite rank $\mathcal{H}[\mathbf{B}]$-spaces

5.1 Finite rank spaces. The applicability of Theorem 2.2 depends highly on the ability to identify the spaces W and $\Theta H^2(Y_1)$ in (8). In general, $\Delta(\zeta)$ defined by (7) is taking values in the algebra of operators on an infinite dimensional Hilbert space. The situation is much more tractable in the case of finite rank $\mathcal{H}[\mathbf{B}]$-spaces, for then $\Delta(\zeta)$ acts on a finite dimensional space. In this last section we restrict ourselves to the study of the finite rank case. Thus, we study spaces of the form $\mathcal{H} = \mathcal{H}[\mathbf{B}]$ with $\mathbf{B} = (b_1, \dots, b_n)$, where n is the rank of $\mathcal{H}[\mathbf{B}]$ which was defined in the introduction. It follows that $b_1, \dots, b_n$ are linearly independent. For convenience, we will also be assuming that $\mathbf{B} \neq 0, 1$, which correspond to the cases $\mathcal{H}[\mathbf{B}] = H^2$ and $\mathcal{H}[\mathbf{B}] = \{0\}$, respectively.

5.2 M_z-invariance and consequences. It turns out that the decomposition (8) is particularly simple in the case when $\mathcal{H}[\mathbf{B}]$ is invariant for M_z.

Proposition 5.1. *If the finite rank $\mathcal{H}[\mathbf{B}]$-space is invariant for M_z, then $W = \{0\}$ in the decomposition* (8)*, and thus Theorem* 2.4 *applies, with $\mathbf{A}$ a square matrix and* $\det \mathbf{A} \neq 0$.

Proof. In the notation of Theorem 2.2 we have $Y \simeq \mathbb{C}^n$, thus (8) becomes

$$\operatorname{clos}(\Delta H^2(\mathbb{C}^n)) = W \oplus \Theta H^2(\mathbb{C}^m), \tag{25}$$

where $m \leq n$. We will see that $W = \{0\}$ by showing something stronger, namely that $m = n$ in (25). To this end, let $f \in \mathcal{H}[\mathbf{B}]$ and consider $Jf = (f, \mathbf{g})$ and $JM_z f = (M_z f, \mathbf{g}_0)$, J being the embedding given by Theorem 2.2. Let $\mathbf{g} = \mathbf{w} + \Theta\mathbf{h}$ and $\mathbf{g}_0 = \mathbf{w}_0 + \Theta\mathbf{h}_0$ be the decompositions with respect to (25). Since $LM_z f = f$, we see from part (ii) of Theorem 2.2 that $\mathbf{w}_0(\zeta) = \zeta\mathbf{w}(\zeta)$ and $\mathbf{h}_0(\zeta) = \zeta\mathbf{h}(\zeta) + \mathbf{c}_f$, where $\mathbf{c}_f \in \mathbb{C}^m$. By part (i) of Theorem 2.2 we have that

$$\begin{aligned} &\mathbf{B}(\zeta)^*\zeta\mathbf{f}(\zeta) + \Delta(\zeta)\mathbf{g}_0(\zeta) \\ &= \zeta\Big(\mathbf{B}(\zeta)^*\mathbf{f}(\zeta) + \Delta(\zeta)\mathbf{g}(\zeta)\Big) + \mathbf{A}(\zeta)^*\mathbf{c}_f \in \overline{H_0^2(\mathbb{C}^n)}, \end{aligned} \tag{26}$$

where, as before, $\mathbf{A} = \Theta^*\Delta$. We apply this to the reproducing kernel $k_\lambda(z)$ of $\mathcal{H}[\mathbf{B}]$. Recall from Section 2 that

$$Jk_\lambda = \left(\frac{1 - \mathbf{B}(z)\mathbf{B}(\lambda)^*}{1 - \overline{\lambda}z}, \frac{-\Delta(\zeta)\mathbf{B}(\lambda)^*}{1 - \overline{\lambda}z} \right) = (k_\lambda, \mathbf{g}_\lambda).$$

A brief computations shows that

$$\mathbf{B}^*(\zeta)k_\lambda(\zeta) + \Delta(\zeta)\mathbf{g}_\lambda(\zeta) = \overline{\zeta}\frac{\mathbf{B}(\zeta)^* - \mathbf{B}(\lambda)^*}{\overline{\zeta} - \lambda}.$$

Using (26) we deduce that for each $\lambda \in \mathbb{D}$ there exists $\mathbf{c}_\lambda \in \mathbb{C}^m$ such that

$$\frac{\mathbf{B}(\zeta)^* - \mathbf{B}(\lambda)^*}{\overline{\zeta} - \lambda} + \mathbf{A}(\zeta)^*\mathbf{c}_\lambda \in \overline{H_0^2(\mathbb{C}^n)}.$$

Since $\mathbf{B}(0) = 0$ we see that the constant term of the above function equals

$$0 = \mathbf{B}(\lambda)^*/\lambda + \mathbf{A}(0)^*\mathbf{c}_\lambda.$$

By linear independence of the coordinates $\{b_i\}_{i=1}^n$ we conclude that $\mathbf{A}(0)^*$ maps an m-dimensional vector space onto an n-dimensional vector space. Thus it follows that $m = n$, and hence $W = \{0\}$ in (25). Since $\mathbf{A}$ is outer, invertibility of $\mathbf{A}(0)$ implies that $\det \mathbf{A} \neq 0$, and thus the function $\det \mathbf{A}$ is non-zero and outer. □

The following result characterizes M_z-invariance in terms of modulus of $\mathbf{B}$, and is a generalization of a well-known theorem for $\mathcal{H}(b)$-spaces.

Theorem 5.2. *A finite rank $\mathcal{H}[\mathbf{B}]$-space is invariant under the forward shift operator M_z if and only if*

$$\int_{\mathbb{T}} \log(1 - \|\mathbf{B}\|_2^2)dm = \int_{\mathbb{T}} \log(1 - \sum_{i=1}^n |b_i|^2)dm > -\infty.$$

Proof. Assume that $\mathcal{H}[\mathbf{B}]$ is M_z-invariant. Then $\det \mathbf{A}(z)$ is non-zero by Proposition 5.1. In terms of boundary values on $\mathbb{T}$ we have $\mathbf{A} = \Theta^*\Delta = \Theta^*(I_n - \mathbf{B}^*\mathbf{B})^{1/2}$, where I_n is the n-by-n identity matrix, and so $\det \mathbf{A} = \overline{\det \Theta}(1 - \sum_{i=1}^n |b_i|^2)^{1/2}$ on $\mathbb{T}$. Since Θ is an isometry we have that $|\det \Theta| = 1$, and thus

$$\int_{\mathbb{T}} \log(1 - \sum_{i=1}^n |b_i|^2)dm = \int_{\mathbb{T}} 2\log(|\det \mathbf{A}|)dm > -\infty,$$

last inequality being a well-known fact for bounded analytic functions.

Conversely, assume that $\int_{\mathbb{T}} \log(1 - \sum_{i=1}^n |b_i|^2)dm > -\infty$. Thus $1 - \sum_{i=1}^n |b_i|^2 = \det \Delta^2 > 0$ almost everywhere on $\mathbb{T}$. Consider again the decomposition in (25). We claim that $W = \{0\}$. Assume, seeking a contradiction, that $W \neq \{0\}$. W is invariant under multiplication by scalar-valued bounded measurable functions, and so W contains a function $\mathbf{g}$ which is non-zero but vanishes on a set of positive measure. Fix $\mathbf{h}_n \in H^2(\mathbb{C}^n)$ such that $\Delta\mathbf{h}_n \to \mathbf{g}$. Let $\text{adj}(\Delta) := (\det \Delta)\Delta^{-1}$ be the adjugate matrix. Then $\text{adj}(\Delta)$ has bounded entries and thus

$$\det \Delta \mathbf{h}_n = \text{adj}(\Delta)\Delta\mathbf{h}_n \to \text{adj}(\Delta)\mathbf{g}, \tag{27}$$

in $L^2(\mathbb{C}^n)$. By our assumption, there exists an analytic outer function d with $|d| = |\det \Delta| = (1 - \sum_{i=1}^n |b_i|^2)^{1/2}$ almost everywhere on $\mathbb{T}$. Then

$\det \Delta = \psi d$ for some measurable function ψ of modulus 1 almost everywhere on $\mathbb{T}$, and $\det \Delta \mathbf{h}_n = \psi d \mathbf{h}_n \in \psi H^2(\mathbb{C}^n)$, where $\psi H^2(\mathbb{C}^n)$ is norm-closed and contains no non-zero function which vanishes on a set of positive measure on $\mathbb{T}$. But by (27) we have $\operatorname{adj}(\Delta)\mathbf{g} \in \psi H^2(\mathbb{C}^n)$, which is a contradiction. Thus $W = \{0\}$ in (25), so that $\operatorname{clos}(\Delta H^2(\mathbb{C}^n)) = \Theta H^2(\mathbb{C}^m)$ and Theorem 2.4 applies. We have $1 - \sum_{i=1}^n |b_i(\zeta)|^2 > 0$ almost everywhere on $\mathbb{T}$, and therefore $\Delta(\zeta)$ is invertible almost everywhere on $\mathbb{T}$. This implies that $m = n$. Since $\mathbf{A}(\zeta) = \Theta(\zeta)^* \Delta(\zeta)$, we see that $\mathbf{A}(z)$ is an n-by-n matrix-valued outer function. In particular, $\mathbf{A}(z)$ is invertible at every $z \in \mathbb{D}$. Let J be the embedding of Theorem 2.4 and $Jf = (f, \mathbf{f}_1)$, where $f \in \mathcal{H}[\mathbf{B}]$ is arbitrary. We claim that we can find a vector $\mathbf{c}_f \in \mathbb{C}^n$ such that

$$\mathbf{B}(\zeta)^* \zeta f(\zeta) + \mathbf{A}(\zeta)^* (\zeta \mathbf{f}_1(\zeta) + \mathbf{c}_f) \in \overline{H_0^2(\mathbb{C}^n)}. \tag{28}$$

Indeed, by (i) of Theorem 2.4 we have that $\mathbf{B}(\zeta)^* f + \mathbf{A}(\zeta)^* \mathbf{f}_1(\zeta) \in \overline{H_0^2(\mathbb{C}^n)}$. Let $\mathbf{v}$ be the zeroth-order coefficient of the coanalytic function $\zeta \mathbf{B}(\zeta)^* f + \zeta \mathbf{A}(\zeta)^* \mathbf{f}_1(\zeta)$. It then suffices to take $\mathbf{c}_f = -(\mathbf{A}(0)^*)^{-1} \mathbf{v}$ for (28) to hold. Thus by (i) of Theorem 2.4 we have that $zf(z) \in \mathcal{H}[\mathbf{B}]$, which completes the proof. □

From now on we will be working exclusively with M_z-invariant $\mathcal{H}[\mathbf{B}]$-spaces of finite rank, and J will always denote the embedding of $\mathcal{H}[\mathbf{B}]$-space that appears in Theorem 2.4.

Lemma 5.3. *If a finite rank $\mathcal{H}[\mathbf{B}]$-space is M_z-invariant, then the spectral radius of the operator M_z equals* 1.

Proof. We have seen above that if $Jf = (f, \mathbf{f}_1)$, then $JM_z f = (zf(z), z\mathbf{f}_1(z) + \mathbf{c}_f)$, where $\mathbf{c}_f \in \mathbb{C}^n$ is some vector depending on f. This shows that M_z is unitarily equivalent to a finite rank perturbation of the isometric shift operator $(g, \mathbf{g}_1) \mapsto (zg, z\mathbf{g}_1)$ acting on $H^2 \oplus H^2(\mathbb{C}^n)$. It follows that the essential spectra of the two operators coincide, and so are contained in $\operatorname{clos}(\mathbb{D})$. Thus for $|\lambda| > 1$ the operator $M_z - \lambda$ is Fredholm of index 0. Since $M_z - \lambda$ is injective, index 0 implies that that $M_z - \lambda$ is invertible. □

A consequence of Lemma 5.3 is that the operators $(I_{\mathcal{H}[\mathbf{B}]} - \overline{\lambda} M_z)^{-1}$ exist for $|\lambda| < 1$. Their action is given by

$$(I_{\mathcal{H}[\mathbf{B}]} - \overline{\lambda} M_z)^{-1} f(z) = \frac{f(z)}{1 - \overline{\lambda} z}.$$

Let

$$J \frac{f(z)}{1 - \overline{\lambda} z} = \Big(\frac{f(z)}{1 - \overline{\lambda} z}, \mathbf{g}_\lambda(z) \Big)$$

for some $\mathbf{g}_\lambda \in H^2(\mathbb{C}^n)$. We compute

$$Jf = J(I_{\mathcal{H}[\mathbf{B}]} - \overline{\lambda} M_z)(I_{\mathcal{H}[\mathbf{B}]} - \overline{\lambda} M_z)^{-1} f = \Big(f, (1 - \overline{\lambda} z)\mathbf{g}_\lambda(z) + \mathbf{c}_f(\lambda) \Big),$$

where $\mathbf{c}_f(\lambda) \in \mathbb{C}^n$ is some vector depending on f and λ. By re-arranging we obtain $\mathbf{g}_\lambda(z) = \frac{\mathbf{f}_1(z)-\mathbf{c}_f(\lambda)}{1-\overline{\lambda}z}$, and so

$$J\frac{f(z)}{1-\overline{\lambda}z} = \Big(\frac{f(z)}{1-\overline{\lambda}z}, \frac{\mathbf{f}_1(z) - \mathbf{c}_f(\lambda)}{1-\overline{\lambda}z}\Big). \tag{29}$$

We will now show that $\mathbf{c}_f(\lambda)$ is actually a coanalytic function of λ which admits non-tangential boundary values almost everywhere on $\mathbb{T}$. To this end, let $\mathbf{u}_f(z) \in \overline{H^2_0(\mathbb{C}^n)}$ be the coanalytic function with boundary values

$$\mathbf{u}_f(\zeta) = \mathbf{B}(\zeta)^* f(\zeta) + \mathbf{A}(\zeta)^*\mathbf{f}_1(\zeta).$$

By (i) of Theorem 2.4 and (29) we have

$$\frac{\mathbf{B}(\zeta)^* f(\zeta) + \mathbf{A}(\zeta)^*\mathbf{f}_1(\zeta)}{1-\overline{\lambda}\zeta} - \frac{\mathbf{A}(\zeta)^*\mathbf{c}_f(\lambda)}{1-\overline{\lambda}\zeta} = \frac{\mathbf{u}_f(\zeta)}{1-\overline{\lambda}\zeta} - \frac{\mathbf{A}(\zeta)^*\mathbf{c}_f(\lambda)}{1-\overline{\lambda}\zeta} \in \overline{H^2_0(\mathbb{C}^n)}, \tag{30}$$

and projecting this equation onto $H^2(\mathbb{C}^n)$ we obtain

$$\mathbf{u}_f(\lambda) = A(\lambda)^*\mathbf{c}_f(\lambda). \tag{31}$$

Since $\mathbf{A}(\lambda)^*$ is coanalytic and invertible for every $\lambda \in \mathbb{D}$, we get that

$$\mathbf{c}_f(\lambda) = (\mathbf{A}(\lambda)^*)^{-1}\mathbf{u}_f(\lambda) = \frac{\text{adj}(\mathbf{A}(\lambda)^*)\mathbf{u}_f(\lambda)}{\overline{\det \mathbf{A}(\lambda)}}$$

is coanalytic. Moreover, the last expression shows that $\mathbf{c}_f(\lambda)$ is in the coanalytic Smirnov class $\overline{N^+(\mathbb{C}^n)}$, and thus admits non-tangential limits almost everywhere on $\mathbb{T}$. For the boundary function we have the equality

$$\mathbf{c}_f(\zeta) = (\mathbf{A}(\zeta)^*)^{-1}\mathbf{u}_f(\zeta) = (\mathbf{A}(\zeta)^*)^{-1}\mathbf{B}(\zeta)^* f(\zeta) + \mathbf{f}_1(\zeta) \tag{32}$$

almost everywhere on $\mathbb{T}$. The following proposition summarizes the discussion above.

Proposition 5.4. *Let $\mathcal{H}[\mathbf{B}]$ be of finite rank and M_z-invariant. If $f \in \mathcal{H}[\mathbf{B}]$, then for all $\lambda \in \mathbb{D}$ we have $\frac{f(z)}{1-\overline{\lambda}z} \in \mathcal{H}[\mathbf{B}]$ and*

$$J\frac{f(z)}{1-\overline{\lambda}z} = \Big(\frac{f(z)}{1-\overline{\lambda}z}, \frac{\mathbf{f}_1(z) - \mathbf{c}_f(\lambda)}{1-\overline{\lambda}z}\Big),$$

where $\mathbf{c}_f$ is a coanalytic function of λ which admits non-tangential boundary values almost everywhere on $\mathbb{T}$, and (32) *holds.*

5.3 **Density of polynomials.** Since we are assuming that $1 \in \mathcal{H}[\mathbf{B}]$, the M_z-invariance implies that the polynomials are contained in the space. The following result generalizes a well-known polynomial density result for $\mathcal{H}(b)$-spaces (see Section IV-3 of [30]).

Theorem 5.5. *If finite rank $\mathcal{H}[\mathbf{B}]$ is M_z-invariant, then the polynomials are dense in $\mathcal{H}$.*

Proof. Since $J1 = (1, 0)$, an application of Proposition 5.4 and (32) shows that

$$J\frac{1}{1-\overline{\lambda}z} = \left(\frac{1}{1-\overline{\lambda}z}, -\frac{(\mathbf{A}(\lambda)^*)^{-1}\mathbf{B}(\lambda)^*1}{1-\overline{\lambda}z}\right), \quad \lambda \in \mathbb{D} \tag{33}$$

Assume that $f \in \mathcal{H}[\mathbf{B}]$ is orthogonal to the polynomials. Then it follows that $Jf = (f, \mathbf{f}_1)$ is orthogonal in $H^2 \oplus H^2(\mathbb{C}^n)$ to tuples of the form given by (33). Thus

$$\begin{aligned} 0 &= f(\lambda) - \left\langle \mathbf{f}_1(\lambda), (\mathbf{A}(\lambda)^*)^{-1}\mathbf{B}(\lambda)^*1 \right\rangle_{\mathbb{C}^n} \\ &= f(\lambda) - \mathbf{B}(\lambda)(\mathbf{A}(\lambda))^{-1}\mathbf{f}_1(\lambda). \end{aligned}$$

Set

$$\mathbf{h}(z) = (\mathbf{A}(z))^{-1}\mathbf{f}_1(z) = \frac{\operatorname{adj}(\mathbf{A}(z))\mathbf{f}_1(z)}{\det \mathbf{A}(z)}.$$

The last expression shows that $\mathbf{h}$ belongs to the Smirnov class $N^+(\mathbb{C}^n)$. By the Smirnov maximum principle we furthermore have $\mathbf{h} \in H^2(\mathbb{C}^n)$, since

$$\begin{aligned} \int_{\mathbb{T}} \|\mathbf{h}\|^2_{\mathbb{C}^n} dm(\zeta) &= \int_{\mathbb{T}} \|\mathbf{B}\mathbf{h}\|^2_{\mathbb{C}^n} dm + \int_{\mathbb{T}} \|\mathbf{A}\mathbf{h}\|^2_{\mathbb{C}^n} dm \\ &= \int_{\mathbb{T}} |f|^2 dm + \int_{\mathbb{T}} \|\mathbf{f}_1\|^2_{\mathbb{C}^n} dm = \|f\|^2_{\mathcal{H}[\mathbf{B}]} < \infty. \end{aligned}$$

Hence $Jf = (f, \mathbf{f}_1) = (\mathbf{B}\mathbf{h}, \mathbf{A}\mathbf{h})$ with $\mathbf{h} \in H^2(\mathbb{C}^n)$, so $Jf \in (J\mathcal{H}[\mathbf{B}])^\perp$ and it follows that $f = 0$. □

5.4 **Equivalent norms.** In the finite rank case we meet a natural but fundamental question whether the rank of an $\mathcal{H}[\mathbf{B}]$-space can be reduced with help of an equivalent norm. Here we assume that the space satisfies (A.1)-(A.3) with respect to the new norm. If that is the case, then the renormed space is itself an $\mathcal{H}[\mathbf{D}]$-space for some function $\mathbf{D}$. We will say that the spaces $\mathcal{H}[\mathbf{B}]$ and $\mathcal{H}[\mathbf{D}]$ are *equivalent* if they are equal as sets and the two induced norms are equivalent. As mentioned in the introduction, [14] shows that in the case $\mathcal{H}[\mathbf{B}] = \mathcal{D}(\mu)$ for μ a finite set of point masses the space is equivalent to a rank one $\mathcal{H}(b)$-space.

It is relatively easy to construct a $\mathbf{B} = (b_1, b_2)$ such that $\mathcal{H}[\mathbf{B}] = \mathcal{H}(b) \cap K_\theta$, where b is non-extreme and $K_\theta = H^2 \ominus \theta H^2$. For appropriate choices of b and θ, the space $\mathcal{H}(b) \cap K_\theta$ cannot be equivalent to a de Branges-Rovnyak space. We shall not go into details of the construction, but we mention that it can be deduced from the proof of Theorem 5.12. However, it is less obvious how to obtain an example of an $\mathcal{H}[\mathbf{B}]$-space which is M_z-invariant and which is not equivalent to a non-extreme de Branges-Rovnyak space. The purpose of this section is to verify existence of such a space, and thus confirm that the class of M_z-invariant $\mathcal{H}[\mathbf{B}]$-spaces is indeed a non-trivial extension of the $\mathcal{H}(b)$-spaces constructed from non-extreme b. The result that we shall prove is the following.

Theorem 5.6. *For each integer $n \geq 1$ there exists $\mathbf{B} = (b_1, \ldots, b_n)$ such that $\mathcal{H}[\mathbf{B}]$ is M_z-invariant and has the following property: if $\mathcal{H}[\mathbf{B}]$ is equivalent to a space $\mathcal{H}[\mathbf{D}]$ with $\mathbf{D} = (d_1, \ldots, d_m)$, then $m \geq n$.*

Our first result in the direction of the proof of Theorem 5.6 is interesting in its own right and gives a criterion for when two spaces $\mathcal{H}[\mathbf{B}]$ and $\mathcal{H}[\mathbf{D}]$ are equivalent. Below, $\overline{H^\infty}$ and $\overline{N^+}$ will denote the spaces of complex conjugates of functions in H^∞ and N^+, respectively.

Theorem 5.7. *Let $\mathcal{H}[\mathbf{B}]$ with $\mathbf{B} = (b_1, \ldots, b_n)$ and $\mathcal{H}[\mathbf{D}]$ with $\mathbf{D} = (d_1, \ldots, d_m)$ be two spaces with embeddings $J_1 : \mathcal{H}[\mathbf{B}] \to H^2 \oplus H^2(\mathbb{C}^n)$ and $J_2 : \mathcal{H}[\mathbf{D}] \to H^2 \oplus H^2(\mathbb{C}^m)$ as in Theorem 2.4, such that*

$$J_1 \frac{1}{1-\overline{\lambda}z} = \frac{(1, \mathbf{c}(\lambda))}{1-\overline{\lambda}z} = \frac{(1, c_1(\lambda), \ldots, c_n(\lambda))}{1-\overline{\lambda}z},$$
$$J_2 \frac{1}{1-\overline{\lambda}z} = \frac{(1, \mathbf{e}(\lambda))}{1-\overline{\lambda}z} = \frac{(1, e_1(\lambda), \ldots, e_m(\lambda))}{1-\overline{\lambda}z}.$$

Then the spaces $\mathcal{H}[\mathbf{B}]$ and $\mathcal{H}[\mathbf{D}]$ are equivalent if and only if the $\overline{H^\infty}$-submodules of $\overline{N^+}$ generated by $\{1, c_1, \ldots, c_n\}$ and by $\{1, e_1, \ldots, e_m\}$ coincide.

Proof. Assume that $\mathcal{H}[\mathbf{B}]$ and $\mathcal{H}[\mathbf{D}]$ are equivalent. Let $i : \mathcal{H}[\mathbf{B}] \to \mathcal{H}[\mathbf{D}]$ be the identity mapping $if = f$. The subspaces $K_1 = J_1\mathcal{H}[\mathbf{B}]$ and $K_2 = J_2\mathcal{H}[\mathbf{D}]$ are invariant under the backward shift L by Theorem 2.4, and if $T = J_2 i J_1^{-1} : K_1 \to K_2$, then $LT = TL$. The commutant lifting theorem (see Theorem 10.29 and Exercise 10.31 of [1]) implies that T extends to a Toeplitz operator $T_{\mathbf{\Phi}}$, where $\mathbf{\Phi}$ is an $(m+1)$-by-$(n+1)$ matrix of bounded coanalytic functions, acting by matrix multiplication followed by the component-wise projection P_+ from L^2 onto H^2:

$$T_{\mathbf{\Phi}}(f, f_1, \ldots, f_n)^t = P_+ \mathbf{\Phi}(f, f_1, \ldots, f_n)^t.$$

Thus we have

$$\frac{(1, \mathbf{e}(\lambda))^t}{1-\overline{\lambda}z} = T_{\mathbf{\Phi}} \frac{(1, \mathbf{c}(\lambda))^t}{1-\overline{\lambda}z} = \frac{\mathbf{\Phi}(\lambda)(1, \mathbf{c}(\lambda))^t}{1-\overline{\lambda}z}. \tag{34}$$

By reversing the roles of $\mathcal{H}[\mathbf{B}]$ and $\mathcal{H}[\mathbf{D}]$, we obtain again a coanalytic Toeplitz operator $T_{\mathbf{\Psi}}$ such that

$$\frac{(1, \mathbf{c}(\lambda))^t}{1-\overline{\lambda}z} = T_{\mathbf{\Psi}} \frac{(1, \mathbf{e}(\lambda))^t}{1-\overline{\lambda}z} = \frac{\mathbf{\Psi}(\lambda)(1, \mathbf{e}(\lambda))^t}{1-\overline{\lambda}z}. \tag{35}$$

The equations (34) and (35) show that the $\overline{H^\infty}$-submodule of $\overline{N^+}$ generated by $\{1, c_1, \dots, c_n\}$ coincides with the $\overline{H^\infty}$-submodule generated by $\{1, e_1, \dots, e_m\}$.

Conversely, assume that the $\overline{H^\infty}$-submodules of $\overline{N^+}$ generated by $\{1, c_1, \dots, c_n\}$ and $\{1, e_1, \dots, e_m\}$ coincide. Then there exists a matrix of coanalytic bounded functions $\mathbf{\Phi}$ such that

$$\mathbf{\Phi}(1, c_1, \dots, c_n)^t = (1, e_1, \dots, e_m)^t$$

where we can choose the top row of $\mathbf{\Phi}$ to equal $(1, 0, \dots, 0)$, and there exists also a matrix of coanalytic bounded functions $\mathbf{\Psi}$ such that

$$\mathbf{\Psi}(1, e_1, \dots, e_m)^t = (1, c_1, \dots, c_n)^t$$

with top row $(1, 0, \dots, 0)$. It is now easy to see by the density of the linear span of $\frac{1}{1-\overline{\lambda}z}$ in the spaces $\mathcal{H}[\mathbf{B}]$ and $\mathcal{H}[\mathbf{D}]$ (which can be deduced from Theorem 5.5 and Lemma 5.3) that, in the notation of the above paragraph, we have $T_{\mathbf{\Phi}} K_1 = K_2$, $T_{\mathbf{\Psi}} K_2 = K_1$, $T_{\mathbf{\Psi}} T_{\mathbf{\Phi}} = I_{K_1}$, $T_{\mathbf{\Phi}} T_{\mathbf{\Psi}} = I_{K_2}$ and thus $\mathcal{H}[\mathbf{B}]$ and $\mathcal{H}[\mathbf{D}]$ are equivalent.

□

Lemma 5.8. *Let $\mathbf{c} = (c_1, \dots, c_n)$ be an arbitrary n-tuple of functions in $\overline{N^+}$. There exists a $\mathbf{B} = (b_1, \dots, b_n)$ such that $\mathcal{H}[\mathbf{B}]$ is M_z-invariant and an embedding $J : \mathcal{H}[\mathbf{B}] \to H^2 \oplus H^2(\mathbb{C}^n)$ as in Theorem 2.4 such that*

$$J \frac{1}{1-\overline{\lambda}z} = \left(\frac{1}{1-\overline{\lambda}z}, \frac{\mathbf{c}(\lambda)}{1-\overline{\lambda}z} \right).$$

Proof. Since $\overline{c_i} \in N^+$, there exists a factorization $\overline{c_i} = d_i/u_i$, where $d_i, u_i \in H^\infty$, and u_i is outer. Let $\mathbf{D} = (-d_1, \dots, -d_n)$ and $\mathbf{U} = \operatorname{diag}(u_1, \dots, u_n)$. The linear manifold

$$V = \{(\mathbf{D}\mathbf{h}, \mathbf{U}\mathbf{h}) : \mathbf{h} \in H^2(\mathbb{C}^n)\} \subset H^2 \oplus H^2(\mathbb{C}^n)$$

is invariant under the forward shift M_z. It follows from general theory of shifts and the Beurling-Lax theorem (see Chapter 1 of [28]) that M_z acting on $\operatorname{clos}(V)$ is a shift of multiplicity n, and

$$\operatorname{clos}(V) = \{(\mathbf{B}\mathbf{h}, \mathbf{A}\mathbf{h}) : \mathbf{h} \in H^2(\mathbb{C}^n)\}$$

for some analytic $\mathbf{B}(z) = (b_1(z), \dots, b_n(z))$ and n-by-n matrix-valued analytic $\mathbf{A}(z)$ such that the mapping $\mathbf{h} \mapsto (\mathbf{B}\mathbf{h}, \mathbf{A}\mathbf{h})$ is an isometry from $H^2(\mathbb{C}^n)$ to $H^2 \oplus H^2(\mathbb{C}^n)$. It is easy to see that $\mathbf{A}$ must be outer, since $\mathbf{U}$ is, and that $\sum_{i=1}^n |b_i(z)|^2 \le 1$ for all $z \in \mathbb{D}$. If P is the projection from $H^2 \oplus H^2(\mathbb{C}^n)$ onto the first coordinate H^2, then as in the proof of Theorem 2.2 we set $\mathcal{H}[\mathbf{B}]$ to be the image of $\operatorname{clos}(V)^\perp$ under P, $\|f\|_{\mathcal{H}[\mathbf{B}]} = \|P^{-1}f\|_{H^2 \oplus H^2(\mathbb{C}^n)}$ and $J = P^{-1}$. The tuple $\left(\frac{1}{1-\overline{\lambda}z}, \frac{\mathbf{c}(\lambda)}{1-\overline{\lambda}z}\right)$ is obviously orthogonal to V, so $J\frac{1}{1-\overline{\lambda}z} = \left(\frac{1}{1-\overline{\lambda}z}, \frac{\mathbf{c}(\lambda)}{1-\overline{\lambda}z}\right)$. The forward shift invariance of $\mathcal{H}[\mathbf{B}]$ can be seen using the same argument as in the end of the proof of Theorem 5.2. □

The next lemma is a version of a result of R. Mortini, see Lemma 2.8 and Theorem 2.9 of [24]. The proofs in [24] can be readily adapted to prove our version, we include a proof sketch for the convenience of the reader.

Lemma 5.9. *For each $n \geq 1$ there exists an outer function $u \in H^\infty$ and $f_1, \ldots, f_n \in H^\infty$ such that the ideal*

$$\left\{ g_0 u + \sum_{i=1}^{n} g_i f_i : g_0, g_1, \ldots, g_n \in H^\infty \right\}$$

cannot be generated by less than $n+1$ functions in H^∞.

Proof. Let $\mathcal{M}$ be the maximal ideal space of H^∞. The elements of H^∞ are naturally functions on $\mathcal{M}$, and if $\xi \in \mathcal{M}$, then the evaluation of $f \in H^\infty$ at ξ will be denoted by $f(\xi)$. Let u be a bounded outer function and I be an inner function such that (u, I) is not a corona pair, so that there exists $\xi \in M$ such that $u(\xi) = I(\xi) = 0$. Let $f_k = I^k u^{n-k}$. We claim that the ideal generated by $\{u^n, f_1, \ldots, f_n\}$ cannot be generated by less than $n+1$ functions.

The proof is split into two parts. In the first part we apply the idea contained in Lemma 2.8 of [24] to verify the following claim: if $\phi_0, \phi_1, \ldots, \phi_n \in H^\infty$ are such that $\phi_0 u^n + \phi_1 f_1 + \ldots \phi_n f_n = 0$, then $\phi_k(\xi) = 0$ for $0 \leq k \leq n$. To this end, the equality

$$\phi_0 u^n = -(\phi_1 u^{n-1} I + \ldots + \phi_n I^n)$$

shows that ϕ_0 is divisible by I, since the right-hand side is, but u^n, being outer, is not. It follows that $\phi_0 = I h_0$ for some $h_0 \in H^\infty$, and therefore $\phi_0(\xi) = I(\xi) h_0(\xi) = 0$. Dividing the above equality by I and re-arranging, we obtain

$$(h_0 u + \phi_1) u^{n-1} = -(\phi_2 u^{n-2} I + \ldots + \phi_n I^{n-1}).$$

As above, we must have $h_0 u + \phi_1 = h_1 I$, with $h_1 \in H^\infty$. Then

$$\phi_1(\xi) = h_1(\xi) I(\xi) - h_0(\xi) u(\xi) = 0.$$

By repeating the argument we conclude that $\phi_0(\xi) = \phi_1(\xi) = \ldots \phi_n(\xi) = 0$.

The second part of the proof is identical to the proof of Theorem 2.9 in [24]. Assuming that the ideal generated by $\{u^n, f_1, \ldots, f_n\}$ is also generated by $\{e_1, \ldots, e_m\}$, we obtain a matrix $\mathbf{M}$ of size m-by-$(n+1)$ and a matrix $\mathbf{N}$ of size $(n+1)$-by-m, both with entires in H^∞, such that

$$\mathbf{M}(u^n, f_1, \ldots, f_n)^t = (e_1, \ldots, e_m)^t, \quad \mathbf{N}(e_1, \ldots, e_m)^t = (u^n, f_1, \ldots, f_n)^t.$$

Then $(\mathbf{NM} - I_{n+1})(u^n, f_1, \ldots, f_n)^t = \mathbf{0}$, where I_{n+1} is the identity matrix of dimension $n+1$. By the first part of the proof we obtain that $\mathbf{N}(\xi)\mathbf{M}(\xi) = I_{n+1}$, where the evaluation of the matrices at ξ is done entrywise. Then the rank of the matrix $\mathbf{N}(\xi)$ is at least $n+1$, i.e., $m \geq n+1$. □

We are ready to prove the main result of the section.

Proof of Theorem 5.6. By Lemma 5.9 there exist $u, f_1, \ldots, f_n \in H^\infty$, with u outer, such that the ideal of H^∞ generated by $\{u, f_1, \ldots, f_n\}$ is not generated by any set of size less than $n+1$. Let $c_i = \overline{f_i/u} \in \overline{N^+}$ and apply Lemma 5.8 to $\mathbf{c} = (c_1, \ldots, c_n)$ to obtain a $\mathbf{B} = (b_1, \ldots, b_n)$ and a space $\mathcal{H}[\mathbf{B}]$ such that

$$J\frac{1}{1-\overline{\lambda}z} = \left(\frac{1}{1-\overline{\lambda}z}, \frac{\mathbf{c}(\lambda)}{1-\overline{\lambda}z}\right).$$

If $\mathcal{H}[\mathbf{B}]$ is equivalent to $\mathcal{H}[\mathbf{D}]$, where $\mathbf{D} = (d_1, \ldots, d_m)$ and

$$J_2\frac{1}{1-\overline{\lambda}z} = \frac{(1, \mathbf{e}(\lambda))}{1-\overline{\lambda}z} = \frac{(1, e_1(\lambda), \ldots, e_m(\lambda))}{1-\overline{\lambda}z},$$

where J_2 is the embedding associated to $\mathcal{H}[\mathbf{D}]$, then Theorem 5.7 implies that the sets $\{u, f_1, \ldots, f_n\}$ and $\{u, \overline{e_1}u, \ldots, \overline{e_m}u\}$ generate the same ideal in H^∞. Thus $m \geq n$. $\square$

5.5 M_z-invariant subspaces. Our structure theorem for M_z-invariant subspaces will follow easily from Theorem 2.4 after this preliminary lemma.

Lemma 5.10. *Let $\mathcal{H}[\mathbf{B}]$ be of finite rank and M_z-invariant. If $\mathcal{M}$ is an M_z-invariant subspace of $\mathcal{H}[\mathbf{B}]$, then for each $\lambda \in \mathbb{D}$ we have that* $\dim \mathcal{M} \ominus (M_z - \lambda)\mathcal{M} = 1$, *and thus M is nearly invariant.*

Proof. Let $f \in \mathcal{M}, h \in \mathcal{M}^\perp$ and $Jf = (f, \mathbf{f}_1), Jh = (h, \mathbf{h}_1)$. By M_z-invariance of $\mathcal{M}$ we have, in the notation of Proposition 5.4,

$$0 = \left\langle (I_{\mathcal{H}[\mathbf{B}]} - \overline{\lambda}M_z)^{-1} f, h \right\rangle_{\mathcal{H}[\mathbf{B}]}$$

$$= \int_{\mathbb{T}} \frac{f(\zeta)\overline{h(\zeta)} + \langle \mathbf{f}_1(\zeta), \mathbf{h}_1(\zeta)\rangle_{\mathbb{C}^n}}{1-\overline{\lambda}\zeta} dm(\zeta) - \langle \mathbf{c}_f(\lambda), \mathbf{h}_1(\lambda)\rangle_{\mathbb{C}^n}. \tag{36}$$

Let $K_{f,h}$ be the Cauchy transform

$$K_{f,h}(\lambda) = \lambda \int_{\mathbb{T}} \frac{f(\zeta)\overline{h(\zeta)} + \langle \mathbf{f}_1(\zeta), \mathbf{h}_1(\zeta)\rangle_{\mathbb{C}^n}}{\zeta - \lambda} dm(\zeta). \tag{37}$$

Then $K_{f,h}$ is analytic for $\lambda \in \mathbb{D}$ and admits non-tangential boundary values on $\mathbb{T}$. Adding (36) and (37) gives

$$K_{f,h}(\lambda) = \int_{\mathbb{T}} \frac{1-|\lambda|^2}{|\zeta-\lambda|^2}\left(f(\zeta)\overline{h(\zeta)} + \langle \mathbf{f}_1(\zeta), \mathbf{h}_1(\zeta)\rangle_{\mathbb{C}^n}\right) dm(\zeta) - \langle \mathbf{c}_f(\lambda), \mathbf{h}_1(\lambda)\rangle_{\mathbb{C}^n}.$$

By taking the limit $|\lambda| \to 1$ and using basic properties of Poisson integrals we see that, for almost every $\lambda \in \mathbb{T}$, we have the equality

$$\begin{aligned} K_{f,h}(\lambda) &= f(\lambda)\overline{h(\lambda)} + \langle \mathbf{f}_1(\lambda), \mathbf{h}_1(\lambda)\rangle_{\mathbb{C}^n} - \langle \mathbf{c}_f(\lambda), \mathbf{h}_1(\lambda)\rangle_{\mathbb{C}^n} \\ &= f(\lambda)\overline{h(\lambda)} - \langle (\mathbf{A}(\lambda)^*)^{-1}\mathbf{B}(\lambda)^* f(\lambda), \mathbf{h}_1(\lambda)\rangle_{\mathbb{C}^n} \\ &= f(\lambda)\Big(\overline{h(\lambda)} - \langle (\mathbf{A}(\lambda)^*)^{-1}\mathbf{B}(\lambda)^* 1, \mathbf{h}_1(\lambda)\rangle_{\mathbb{C}^n}\Big), \end{aligned}$$

where we used (32) in the computation. The meromorphic function $K_{f,h}/f$ thus depends only on h, and not on f.

Let $f(\lambda) = 0$ for some $\lambda \in \mathbb{D} \setminus \{0\}$ which is not a common zero of $\mathcal{M}$, so that there exists $g \in \mathcal{M}$ with $g(\lambda) \neq 0$. From $K_{f,h}/f = K_{g,h}/g$ we deduce that $K_{f,h}(\lambda)g(\lambda) = K_{g,h}(\lambda)f(\lambda) = 0$, and thus

$$\begin{aligned} 0 = K_{f,h}(\lambda) &= \lambda \int_{\mathbb{T}} \frac{f(\zeta)\overline{h(\zeta)} + \langle \mathbf{f}_1(\zeta), \mathbf{h}_1(\zeta)\rangle_{\mathbb{C}^n}}{z - \lambda} dm(\zeta) \\ &= \lambda \int_{\mathbb{T}} \frac{f(\zeta)\overline{h(\zeta)} + \langle \mathbf{f}_1(\zeta) - \mathbf{f}_1(\lambda), \mathbf{h}_1(\zeta)\rangle_{\mathbb{C}^n}}{z - \lambda} dm(\zeta) \\ &= \lambda \Big\langle \frac{f(z)}{z-\lambda}, h \Big\rangle_{\mathcal{H}[\mathbf{B}]}. \end{aligned}$$

Since $h \in \mathcal{M}^\perp$ is arbitrary, we conclude that $\frac{f(z)}{z-\lambda} \in \mathcal{M}$, and thus $\dim \mathcal{M} \ominus (M_z - \lambda)\mathcal{M} = 1$. The fact that $\dim \mathcal{M} \ominus M_z\mathcal{M} = 1$ holds also for $\lambda = 0$ or a common zero of the functions in $\mathcal{M}$ follows from basic Fredholm theory. The operators $M_z - \lambda$ are injective semi-Fredholm operators, and thus $\lambda \mapsto \dim \mathcal{M} \ominus (M_z - \lambda)\mathcal{M}$ is a constant function in $\mathbb{D}$. □

The following is our main theorem on M_z-invariant subspaces of finite rank $\mathcal{H}[\mathbf{B}]$-spaces.

Theorem 5.11. *Let $\mathcal{H} = \mathcal{H}[\mathbf{B}]$ be of finite rank and M_z-invariant and $\mathcal{M}$ be a closed M_z-invariant subspace of $\mathcal{H}$. Then*

(i) $\dim \mathcal{M} \ominus M_z\mathcal{M} = 1$,

(ii) any non-zero element in $\mathcal{M} \ominus M_z\mathcal{M}$ is a cyclic vector for $M_z|\mathcal{M}$,

(iii) if $\phi \in \mathcal{M} \ominus M_z\mathcal{M}$ is of norm 1, then there exists a space $\mathcal{H}[\mathbf{C}]$ invariant under M_z, where $\mathbf{C} = (c_1, \dots, c_k)$ and $k \leq n$, such that

$$\mathcal{M} = \phi\mathcal{H}[\mathbf{C}]$$

and the mapping $g \mapsto \phi g$ is an isometry from $\mathcal{H}[\mathbf{C}]$ onto $\mathcal{M}$, *(iv) if J is the embedding given by Theorem 2.4, $\phi \in \mathcal{M} \ominus M_z\mathcal{M}$ with $J\phi = (\phi, \boldsymbol{\phi}_1)$, then*

$$\mathcal{M} = \{f \in \mathcal{H}[\mathbf{B}] : \tfrac{f}{\phi} \in H^2, \tfrac{f}{\phi}\boldsymbol{\phi_1} \in H^2(\mathbb{C}^n)\}.$$

Proof. Part (i) has been established in Lemma 5.10 and part (ii) follows from (iii) by Theorem 5.5. It thus suffices to prove parts (iii) and (iv).

We verified in Lemma 5.10 that $\mathcal{M}$ is nearly invariant, and thus norm formula (22) of Proposition 4.5 applies. A computation shows that, in the notation of Proposition 4.5, we have $L^\phi_\lambda f = L_\lambda(f - \frac{f(\lambda)}{\phi(\lambda)}\phi)$, at least when $\phi(\lambda) \neq 0$. Thus if $Jf = (f, \mathbf{f}_1)$ and $J\phi = (\phi, \boldsymbol{\phi}_1)$, then by (22) and part (i) of Theorem 2.6 we obtain that

$$\|f\|^2_{\mathcal{H}[\mathbf{B}]} = \|f/\phi\|^2_{H^2} + \|\mathbf{g}_1\|^2_{H^2(\mathbb{C}^n)} \tag{38}$$

where $\mathbf{g}_1(z) = \mathbf{f}_1(z) - \frac{f(z)}{\phi(z)}\boldsymbol{\phi}_1(z)$. The mapping $If := (f/\phi, \mathbf{g}_1)$ is therefore an isometry from $\mathcal{M}$ into $H^2 \oplus H^2(\mathbb{C}^n)$. The identity $IL^\phi f = (L(f/\phi), L\mathbf{g}_1)$ shows that $I\mathcal{M}$ is a backward shift invariant subspace of $H^2 \oplus H^2(\mathbb{C}^n) \simeq H^2(\mathbb{C}^{n+1})$. Consequently by the Beurling-Lax theorem we have that $(I\mathcal{M})^\perp = \Psi H^2(\mathbb{C}^k)$, for some $(n+1)$-by-k matrix-valued bounded analytic function such that $\Psi(\zeta) : \mathbb{C}^k \to \mathbb{C}^{n+1}$ is an isometry for almost every $\zeta \in \mathbb{T}$. We claim that $k \leq n$. Indeed, in other case Ψ is an $(n+1)$-by-$(n+1)$ square matrix, and hence $\psi(z) = \det \Psi(z)$ is a non-zero inner function. We would then obtain

$$\psi H^2(\mathbb{C}^{n+1}) = \Psi \mathrm{adj}(\Psi) H^2(\mathbb{C}^{n+1}) \subset \Psi H^2(\mathbb{C}^{n+1}) = (I\mathcal{M})^\perp. \tag{39}$$

But since $\mathcal{M}$ is shift invariant, the function $p\phi$ is contained in $\mathcal{M}$ for any polynomial p, and hence for any polynomial p there exists a tuple of the form $(p, \mathbf{g})$ in $I\mathcal{M}$. Together with (39) this shows that the polynomials are orthogonal to ψH^2, so $\psi = 0$ and we arrive at a contradiction.

Decompose the matrix Ψ as

$$\Psi(z) = \begin{bmatrix} \mathbf{C}(z) \\ \mathbf{D}(z) \end{bmatrix}$$

where $\mathbf{C}(z) = (c_1(z), \ldots, c_k(z))$ and $\mathbf{D}(z)$ is an n-by-k matrix. Consider the Hilbert space $\tilde{\mathcal{M}} = \mathcal{M}/\phi = \{f/\phi : f \in \mathcal{M}\}$ with the norm $\|f/\phi\|_{\tilde{\mathcal{M}}} = \|f\|_{\mathcal{H}[\mathbf{B}]}$. By (38), the map

$$\tilde{I}f/\phi := \Big(f/\phi, \mathbf{f}_1 - \tfrac{f}{\phi}\boldsymbol{\phi}_1\Big) \tag{40}$$

is an isometry from $\tilde{\mathcal{M}}$ into $H^2 \oplus H^2(\mathbb{C}^n)$, and

$$(\tilde{I}\tilde{\mathcal{M}})^\perp = \{(\mathbf{Ch}, \mathbf{Dh}) : \mathbf{h} \in H^2(\mathbb{C}^k)\}. \tag{41}$$

The argument of Theorem 2.2 can be used to see that

$$k_{\tilde{\mathcal{M}}}(\lambda, z) = \frac{1 - \sum_{i=1}^k \overline{c_i(\lambda)} c_i(z)}{1 - \overline{\lambda} z}$$

is the reproducing kernel of $\tilde{\mathcal{M}}$. Thus $\tilde{\mathcal{M}} = \mathcal{H}[\mathbf{C}]$, and the proof of part (iii) is complete.

Finally, we prove part (iv). The inclusion of $\mathcal{M}$ in the set given in (iv) has been established in the proof of part (iii) above. On the other hand, assume that $f \in \mathcal{H}[\mathbf{B}]$ is contained in that set. We will show that $(f/\phi, \mathbf{f}_1 - \frac{f}{\phi}\boldsymbol{\phi}_1)$ is orthogonal to the set given in (41), and thus $f/\phi \in \tilde{\mathcal{M}}$, so that $f \in \mathcal{M}$. In order to verify the orthogonality claim, we must show that $\mathbf{C}^*\frac{f}{\phi} + \mathbf{D}^*(\mathbf{f}_1 - \frac{f}{\phi}\boldsymbol{\phi}_1) \in \overline{H^2_0(\mathbb{C}^n)}$. According to Proposition 5.4, we have that

$$J\frac{\phi(z)}{1-\overline{\lambda}z} = \Big(\frac{\phi(z)}{1-\overline{\lambda}z}, \frac{\boldsymbol{\phi}_1(z) - \mathbf{c}_\phi(\lambda)}{1-\overline{\lambda}z}\Big)$$

for some coanalytic function $\mathbf{c}_\phi$ which satisfies $\mathbf{c}_\phi = (\mathbf{A}^*)^{-1}\mathbf{B}^*\phi + \boldsymbol{\phi}_1$ on $\mathbb{T}$. Setting $f(z) = \frac{\phi(z)}{1-\overline{\lambda}z}$ in (40) we obtain from (41) that

$$\Big(\frac{1}{1-\overline{\lambda}z}, -\frac{\mathbf{c}_\phi(\lambda)}{1-\overline{\lambda}z}\Big) \perp \{(\mathbf{Ch}, \mathbf{Dh}) : \mathbf{h} \in H^2(\mathbb{C}^k)\}$$

and then it easily follows that $\mathbf{D}^*(\lambda)\mathbf{c}_\phi(\lambda) = \mathbf{C}^*(\lambda)1$. Using (32) we obtain the boundary value equality

$$\mathbf{C}^* = \mathbf{D}^*(\mathbf{A}^*)^{-1}\mathbf{B}^*\phi + \mathbf{D}^*\boldsymbol{\phi}_1,$$

and thus on $\mathbb{T}$ we have

$$\mathbf{C}^*\tfrac{f}{\phi} + \mathbf{D}^*(\mathbf{f}_1 - \tfrac{f}{\phi}\boldsymbol{\phi}_1) = \mathbf{D}^*(\mathbf{A}^*)^{-1}(\mathbf{B}^*f + \mathbf{A}^*\mathbf{f}_1). \tag{42}$$

Since $f \in \mathcal{H}[\mathbf{B}]$ we have that $\mathbf{B}^*f + \mathbf{A}^*g \in \overline{H^2_0(\mathbb{C}^n)}$ and thus (42) represents square-integrable boundary function of a coanalytic function in the Smirnov class. An appeal to the Smirnov maximum principle completes the proof of (iv). □

5.6 Backward shift invariant subspaces.

The lattice of L-invariant subspaces of $\mathcal{H}[\mathbf{B}]$-spaces is much less complicated than the lattice of M_z-invariant subspaces. The following theorem generalizes Theorem 5 of [29] for $\mathcal{H}(b)$ with non-extreme b. Our method of proof is new, and relies crucially on Theorem 2.4.

Theorem 5.12. *Any proper L-invariant subspace of a M_z-invariant finite rank $\mathcal{H}[\mathbf{B}]$-space is of the form*

$$\mathcal{H}[\mathbf{B}] \cap K_\theta,$$

where θ is an inner function and $K_\theta = H^2 \ominus \theta H^2$.

Proof. Let $J : \mathcal{H}[\mathbf{B}] \to H^2 \oplus H^2(\mathbb{C}^n)$ be the embedding of Theorem 2.4. If $\mathcal{M}$ is an L-invariant subspace of $\mathcal{H}[\mathbf{B}]$, then $J\mathcal{M}$ is an L-invariant subspace of $H^2 \oplus H^2(\mathbb{C}^n)$ by (iii) of Theorem 2.4. Thus $(J\mathcal{M})^\perp$ is an M_z-invariant subspace containing $(J\mathcal{H}[\mathbf{B}])^\perp = \{(\mathbf{Bh}, \mathbf{Ah}) : \mathbf{h} \in H^2(\mathbb{C}^n)\}$. Because M_z acting on $(J\mathcal{H}[\mathbf{B}])^\perp$ is a shift of multiplicity n, the multiplicity of M_z acting on $(J\mathcal{M})^\perp$ is at least n, and since $(J\mathcal{M})^\perp \subset H^2 \oplus H^2(\mathbb{C}^n)$, it is at most $n+1$. We claim that this multiplicity must equal $n+1$. Indeed, if it was equal to n, then it is easy to see that

$$(J\mathcal{M})^\perp = \{(\mathbf{C}\mathbf{h}, \mathbf{D}\mathbf{h}) : \mathbf{h} \in H^2(\mathbb{C}^n)\}$$

for some $\mathbf{C}(z) = (c_1(z), \dots, c_n(z))$ and $\mathbf{D}(z)$ an n-by-n-matrix valued analytic function. The fact that no tuple of the form $(0, \mathbf{g})$ is included in $J\mathcal{M}$ implies that $\mathbf{D}$ is an outer function, and thus $\mathbf{D}(\lambda)$ is an invertible operator for every $\lambda \in \mathbb{D}$. The tuple

$$\Big(\frac{1}{1-\overline{\lambda}z}, -\frac{(\mathbf{D}(\lambda)^*)^{-1}\mathbf{C}(\lambda)^*1}{1-\overline{\lambda}z}\Big)$$

is clearly orthogonal to $(J\mathcal{M})^\perp$, and thus $\frac{1}{1-\overline{\lambda}z} \in \mathcal{M}$ for every $\lambda \in \mathbb{D}$. Then $\mathcal{M} = \mathcal{H}[\mathbf{B}]$ by the proof of Theorem 5.5. We assumed that $\mathcal{M}$ is a proper subspace, and so multiplicity of M_z on $(J\mathcal{M})^\perp$ cannot be n.

Having established that M_z is a shift of multiplicity $n+1$ on $(J\mathcal{M})^\perp$, we conclude that

$$(J\mathcal{M})^\perp = \Psi H^2(\mathbb{C}^{n+1}),$$

where Ψ is an $(n+1)$-by-$(n+1)$ matrix-valued inner function, and $\theta = \det \Psi$ is a non-zero scalar-valued inner function. Note that $\theta H^2(\mathbb{C}^{n+1}) = \Psi \operatorname{adj}(\Psi) H^2(\mathbb{C}^{n+1}) \subseteq \Psi H^2(\mathbb{C}^{n+1})$. Thus if $f \in \mathcal{M}$, then $Jf = (f, \mathbf{f}_1) \perp \Psi H^2(\mathbb{C}^{n+1}) \supseteq \theta H^2(\mathbb{C}^{n+1})$. It follows that $f \in K_\theta$, and thus we have shown that $\mathcal{M} \subseteq \mathcal{H}[\mathbf{B}] \cap K_\theta$. Next, consider the $(n+1)$-by-$(n+1)$ matrix

$$\mathbf{M}(z) = \begin{bmatrix} \mathbf{B}(z) & \theta(z) \\ \mathbf{A}(z) & 0 \end{bmatrix}.$$

Then it is easy to see that $f \in \mathcal{H}[\mathbf{B}] \cap K_\theta$ if and only if $Jf = (f, \mathbf{f}_1) \perp \mathbf{M}(z)\mathbf{h}(z)$ for all $\mathbf{h} \in H^2(\mathbb{C}^{n+1})$. If $\mathbf{M}(z) = \mathbf{I}(z)\mathbf{U}(z)$ is the inner-outer factorization of $\mathbf{M}$ into an $(n+1)$-by-$(n+1)$-matrix valued inner function $\mathbf{I}$ and an $(n+1)$-by-$(n+1)$-matrix valued outer function $\mathbf{U}$, then we also have that

$$f \in \mathcal{H}[\mathbf{B}] \cap K_\theta \text{ if and only if } Jf = (f, \mathbf{f}_1) \perp \mathbf{I}H^2(\mathbb{C}^{n+1}). \tag{43}$$

From the containment $\mathcal{M} \subseteq \mathcal{H}[\mathbf{B}] \cap K_\theta$ we get by taking orthocomplements that $\mathbf{I}H^2(\mathbb{C}^{n+1}) \subseteq \Psi H^2(\mathbb{C}^{n+1})$ and thus there exists a factorization $\mathbf{I} = \Psi\mathbf{J}$, where $\mathbf{J}$ is an $(n+1)$-by-$(n+1)$-matrix valued inner function. Since $-\theta \det \mathbf{A} = \det \mathbf{M} = \det \mathbf{I} \det \mathbf{U}$, we see (by comparing inner and outer factors) that $\det \mathbf{I} = \alpha\theta$, with $\alpha \in \mathbb{T}$, and so $\alpha\theta = \det \mathbf{I} = \det \Psi \det \mathbf{J} = \theta \det \mathbf{J}$. We conclude that $\det \mathbf{J}$ is a constant, and thus $\mathbf{J}$ is a constant unitary matrix. But then $(J\mathcal{M})^\perp = \Psi H^2(\mathbb{C}^{n+1}) = \mathbf{I}H^2(\mathbb{C}^{n+1})$, and so the claim follows by (43). $\square$

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Paper v

Nearly invariant subspaces of de Branges spaces

Bartosz Malman

Abstract

We prove that the nearly invariant subspaces of a de Branges space $\mathcal{H}(E)$ which have no common zeros are precisely of the form an exponential function times a de Branges space $\mathcal{H}(E_0)$.

1 Introduction and the main result

Let $\mathcal{H}$ be a space of analytic functions defined on some domain of the complex plane $\mathbb{C}$. A concept commonly appearing in operator theory and complex function theory is that of *nearly invariance* of $\mathcal{H}$. The space is said to be nearly invariant if the zeros of functions in $\mathcal{H}$ can be divided out without leaving the space. More precisely, if $f \in \mathcal{H}$ and $f(\lambda) = 0$, then $f(z)/(z-\lambda) \in \mathcal{H}$. Nearly invariance is sometimes instead referred to as the division property. More generally, if all functions in the space $\mathcal{H}$ vanish on some subset of the complex plane, then the space $\mathcal{H}$ will be called nearly invariant if zeros of the functions in $\mathcal{H}$ can be divided out as long as they do not belong to the common zero set.

This short article is concerned with the (always norm-closed) nearly invariant subspaces of the de Branges spaces. An entire function E which satisfies the inequality $|E(z)| > |E(\overline{z})|$ for z in the upper half plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is called a de Branges function, and to each such function there exists an associated de Branges space $\mathcal{H}(E)$. Let $H^2(\mathbb{C}^+)$ denote the usual Hardy space of the upper half plane, and for an entire function f define $f^*(z) := \overline{f(\overline{z})}$. The de Branges space $\mathcal{H}(E)$ is the Hilbert space of entire functions f which satisfy the following three properties:

(i) $f/E \in H^2(\mathbb{C}^+)$,

(ii) $f^*/E \in H^2(\mathbb{C}^+)$,

(iii) $\|f\|^2_{\mathcal{H}(E)} := \int_{\mathbb{R}} |f/E|^2 dx < \infty$.

The space $\mathcal{H}(E)$, with norm given by (iii) above, is a reproducing kernel Hilbert space with kernel given by

$$k_E(\lambda, z) = \frac{E(z)\overline{E(\lambda)} - E^*(z)\overline{E^*(\lambda)}}{2\pi i(\overline{\lambda} - z)}.$$

Conversely, any kernel of the above form, with E a de Branges function, will of course be the reproducing kernel of a de Branges space. More background information on this class of spaces can be found in de Branges' monograph [2].

The result that we will be proving here is the following structure theorem for nearly invariant subspaces of de Branges spaces.

Theorem 1.1. *Let $\mathcal{N}$ be a nearly invariant subspace with no common zeros of a de Branges space $\mathcal{H}(E)$. Then there exists a de Branges space $\mathcal{H}(E_0)$ and $\alpha \in \mathbb{R}$ such that*

$$\mathcal{N} = e^{i\alpha z}\mathcal{H}(E_0) = \{e^{i\alpha z} f(z) : f \in \mathcal{H}(E_0)\}.$$

For some special classes of de Branges functions E the above theorem can be refined, and as an application of Theorem 1.1 we shall give a new proof of a result of [1] which characterizes the nearly invariant subspaces of the Paley-Wiener spaces. As usual, for $a > 0$, the Paley-Wiener space PW_a consists of the entire functions F which are the Fourier transforms $F = \hat{f}$ of functions f in $L^2(-a, a)$, with an alternative characterization as the space of entire functions of exponential type at most a which are square-integrable on the real axis. Equipped with the usual L^2-norm computed on the real axis, the space PW_a is a de Branges space corresponding to the function $E(z) = e^{-iaz}$.

Corollary 1.2 ([1]). *Let $\mathcal{N}$ be a nearly invariant subspace with no common zeros of a Paley-Wiener space PW_a. Then there exists an interval $I \subseteq (-a, a)$ such that*

$$\mathcal{N} = \{F = \hat{f} \in PW_a : \textit{supp}\, f \subset I\}.$$

2 Proofs

We shall use the notation already introduced in the previous section. Furthermore, we shall use the concept of the Nevanlinna class of the upper half plane, which is the class of analytic functions in $\mathbb{C}^+$ that can be written as a quotient of two bounded analytic functions in $\mathbb{C}^+$. The lower half plane consisting of $z \in \mathbb{C}$ with negative imaginary part will be denoted by $\mathbb{C}^-$.

Lemma 2.1. *If $\mathcal{N}$ is a nearly invariant subspace with no common zeros of a de Branges space $\mathcal{H}(E)$, then the reproducing kernel $k_{\mathcal{N}}$ of $\mathcal{N}$ is of the form*

$$k_{\mathcal{N}}(\lambda, z) = \frac{F(z)\overline{F(\lambda)} - G(z)\overline{G(\lambda)}}{i(z - \overline{\lambda})}, \tag{1}$$

where the functions F and G are entire, and $F/E, F^*/E, G/E, G^*/E$ are in the Nevanlinna class of the upper half plane.

Proof. The proof is very similar to the proof of [2, Theorem 23]. First note that nearly invariance of $\mathcal{N}$ implies that if $f \in \mathcal{N}$ and f vanishes at a point $\alpha \in \mathbb{C}$, then actually we have that $\frac{z-\overline{\alpha}}{z-\alpha} f(z) \in \mathcal{N}$, since

$$\frac{z-\overline{\alpha}}{z-\alpha} f(z) = f(z) + \frac{\alpha-\overline{\alpha}}{z-\alpha} f(z) \in \mathcal{N}.$$

We will use this observation several times below. Fix $\alpha \in \mathbb{C} \setminus \mathbb{R}$, and let f be a function in $\mathcal{N}$ such that $f(\overline{\alpha}) = 0$. For $\lambda \in \mathbb{C}$, the function $\frac{z-\overline{\alpha}}{z-\alpha}\big(k_{\mathcal{N}}(\lambda, z) - \frac{k_{\mathcal{N}}(\lambda,\alpha)}{k_{\mathcal{N}}(\alpha,\alpha)} k_{\mathcal{N}}(\alpha, z)\big)$ is in $\mathcal{N}$, and we have that

$$\begin{aligned}
&\Big\langle f(z), \frac{z-\overline{\alpha}}{z-\alpha}\big(k_{\mathcal{N}}(\lambda, z) - \frac{k_{\mathcal{N}}(\lambda,\alpha)}{k_{\mathcal{N}}(\alpha,\alpha)} k_{\mathcal{N}}(\alpha, z)\big)\Big\rangle_{\mathcal{H}(E)} \\
&= \Big\langle \frac{z-\alpha}{z-\overline{\alpha}} f(z), k_{\mathcal{N}}(\lambda, z) - \frac{k_{\mathcal{N}}(\lambda,\alpha)}{k_{\mathcal{N}}(\alpha,\alpha)} k_{\mathcal{N}}(\alpha, z)\Big\rangle_{\mathcal{H}(E)} \\
&= \frac{\lambda-\alpha}{\lambda-\overline{\alpha}} f(\lambda) = \Big\langle f(z), \frac{\overline{\lambda}-\overline{\alpha}}{\overline{\lambda}-\alpha} k_{\mathcal{N}}(\lambda, z)\Big\rangle_{\mathcal{H}(E)}.
\end{aligned}$$

The function

$$\frac{z-\overline{\alpha}}{z-\alpha}\big(k_{\mathcal{N}}(\lambda, z) - \frac{k_{\mathcal{N}}(\lambda,\alpha)}{k_{\mathcal{N}}(\alpha,\alpha)} k_{\mathcal{N}}(\alpha, z)\big) - \frac{\overline{\lambda}-\overline{\alpha}}{\overline{\lambda}-\alpha} k_{\mathcal{N}}(\lambda, z) \in \mathcal{N}$$

is thus orthogonal to any function in $\mathcal{N}$ which vanishes at $\overline{\alpha}$, and is thus a scalar multiple of $k_{\mathcal{N}}(\overline{\alpha}, z)$. Evaluation at $z = \overline{\alpha}$ shows that

$$\begin{aligned}
&\frac{z-\overline{\alpha}}{z-\alpha}\left(k_{\mathcal{N}}(\lambda, z) - \frac{k_{\mathcal{N}}(\lambda,\alpha)}{k_{\mathcal{N}}(\alpha,\alpha)} k_{\mathcal{N}}(\alpha, z)\right) - \frac{\overline{\lambda}-\overline{\alpha}}{\overline{\lambda}-\alpha} k_{\mathcal{N}}(\lambda, z) \\
&\qquad = -\frac{\overline{\lambda}-\overline{\alpha}}{\overline{\lambda}-\alpha} \cdot \frac{k_{\mathcal{N}}(\lambda,\overline{\alpha})}{k_{\mathcal{N}}(\overline{\alpha},\overline{\alpha})} k_{\mathcal{N}}(\overline{\alpha}, z)
\end{aligned}$$

from which we can solve for $k_{\mathcal{N}}(\lambda, z)$ to obtain that

$$\begin{aligned}
k_{\mathcal{N}}(\lambda, z) = \frac{1}{(\overline{\alpha}-\alpha)(z-\overline{\lambda})} \Bigg(&\frac{k_{\mathcal{N}}(\lambda,\alpha)}{k_{\mathcal{N}}(\alpha,\alpha)} (z-\overline{\alpha})(\overline{\lambda}-\alpha) k_{\mathcal{N}}(\alpha, z) \\
&- \frac{k_{\mathcal{N}}(\lambda,\overline{\alpha})}{k_{\mathcal{N}}(\overline{\alpha},\overline{\alpha})} (z-\alpha)(\overline{\lambda}-\overline{\alpha}) k_{\mathcal{N}}(\overline{\alpha}, z) \Bigg).
\end{aligned}$$

By setting $\alpha = -i/2$,

$$F(z) = k_{\mathcal{N}}(\alpha,\alpha)^{-1/2} k_{\mathcal{N}}(\alpha, z)(z-\overline{\alpha})$$

and

$$G(z) = k_{\mathcal{N}}(\overline{\alpha}, \overline{\alpha})^{-1/2} k_{\mathcal{N}}(\overline{\alpha}, z)(z - \alpha)$$

we see that the kernel $k_{\mathcal{N}}(\lambda, z)$ is of the form as suggested in (1). Moreover, since $k_N(\alpha, z) \in \mathcal{H}(E)$, we have that $k_N(\alpha, z)/E(z) \in H^2(\mathbb{C}^+)$, and so it follows from the above expression for F that F/E is in the Nevanlinna class of the upper half plane. The same is clearly true for $F^*/E, G/E$ and G^*/E. □

Remark. The hypothesis of no common zero set for functions in $\mathcal{N}$ is not used in the proof of the above result. It is however utilized in the following lemmas.

Lemma 2.2. *Let F, G be the entire functions in the expression for the reproducing kernel of $\mathcal{N}$ in* (1). *Then the following properties hold:*

(i) $F^*F = G^*G$,

(ii) $|F(z)| > |G(z)|$ *if* $z \in \mathbb{C}^-$,

(iii) $|F(z)| < |G(z)|$ *if* $z \in \mathbb{C}^+$.

Proof. The function $k_{\mathcal{N}}(\lambda, z)$ is of course an entire function of z, and setting $z = \overline{\lambda}$ we see from (1) that $F(\overline{\lambda})\overline{F(\lambda)} - G(\overline{\lambda})\overline{G(\lambda)} = 0$. This establishes (i). Properties (ii) and (iii) follow from setting $z = \lambda$ and the fact that $\mathcal{N}$ has no common zeros, so that $k_{\mathcal{N}}(\lambda, z)$ is not the zero function, and thus

$$0 < \|k_{\mathcal{N}}(\lambda, \cdot)\|^2_{\mathcal{H}(E)} = k_{\mathcal{N}}(\lambda, \lambda) = \frac{|F(\lambda)|^2 - |G(\lambda)|^2}{-2 \operatorname{Im} \lambda}.$$

□

Lemma 2.3. *Let F, G be the entire functions in the expression for the reproducing kernel of $\mathcal{N}$ in* (1), *and set $U = G^*/F$. Then U is an exponential function, i.e., $U(z) = e^{i\alpha z}$ for some $\alpha \in \mathbb{R}$.*

Proof. The lemma will be established by verifying a series of claims:

(a) U has no zeros and no poles in $\mathbb{C}$, and is thus an entire function,

(b) $|U(x)| = 1$ for every $x \in \mathbb{R}$,

(c) U is in the Nevanlinna class of the upper half plane.

If the above three claims are established, then the fact that U is an exponential function follows easily from the Nevanlinna factorization (see, for instance, [2, Theorem 9 and Problem 27]). We will now establish the three stated claims. Assume that $G^*(\lambda) = \overline{G(\overline{\lambda})} = 0$, and additionally that λ does not lie on the real axis. We will show that F has a zero at λ, of the same order as G^*. Since $|G(\overline{\lambda})| = 0$ we see from (iii) of Lemma 2.2 that $\overline{\lambda}$ must be in the lower half-plane. From (i) of Lemma 2.2 it follows that either F or F^* has a zero at λ, but from (ii) of Lemma 2.2 we see that $|F^*(\lambda)| = |F(\overline{\lambda})| > |G(\overline{\lambda})|$, so $F^*(\lambda)$ is non-zero. Consequently F has a zero at λ, of the same order as G^*. Thus $U = G^*/F$ has no zeros outside of the real axis. In the same manner we can show that U has no poles outside of the real axis.

If x is on the real axis, then by (iii) of Lemma 2.2 we have

$$|F(x)| = \lim_{y \to 0, y>0} |F(x+iy)| \leq \lim_{y \to 0, y>0} |G(x+iy)| = |G(x)|.$$

By considering the limit when $y < 0$ in a similar manner we obtain that $|F(x)| = |G(x)|$ for real x, so that $|U(x)| = 1$, and thus U has no zeros (or poles) on the real axis. This complets the proof of claim (a) and (b). Claim (c) follows from the fact that U is a quotient of G^*/E and F/E, which are in the Nevanlinna class of the upper half plane by Lemma 2.1. □

Lemma 2.4. *Let $\mathcal{N}$ be a nearly invariant subspace with no common zeros of a de Branges space $\mathcal{H}(E)$, with kernel given by* (1), *$U(z) = e^{i\alpha z}$ as in Lemma* 2.3, *and $\sqrt{U}(z) = e^{i\alpha z/2}$. Consider the space*

$$\mathcal{H}_0 = \sqrt{U}\mathcal{N} = \{\sqrt{U}(z) f(z) : f(z) \in \mathcal{N}\}.$$

If $\mathcal{H}_0$ is normed by

$$\|\sqrt{U} f\|_{\mathcal{H}_0} = \|f\|_{\mathcal{H}(E)},$$

then $\mathcal{H}_0$ is a de Branges space.

Proof. It is easy to see from the definition of $\mathcal{H}_0$ and Lemma 2.1 that the reproducing kernel $k_{\mathcal{H}_0}(\lambda, z)$ of the space is

$$k_{\mathcal{H}_0}(\lambda, z) = \overline{\sqrt{U}(\lambda)} \sqrt{U}(z) k_{\mathcal{N}}(\lambda, z)$$
$$= \frac{F(z)\sqrt{U}(z)\overline{F(\lambda)\sqrt{U}(\lambda)} - G(z)\sqrt{U}(z)\overline{G(\lambda)\sqrt{U}(\lambda)}}{i(z - \overline{\lambda})}.$$

Thus $\mathcal{H}_0$ will be a de Branges space if we can show that $(F\sqrt{U})^* = G\sqrt{U}$ and $|F(z)\sqrt{U}(z)| < |F(\overline{z})\sqrt{U}(\overline{z})|$ for $z \in \mathbb{C}^+$. The first claim follows easily from the equality $F^*F = G^*G$, which implies that $F^* = GU$. Indeed, we also have that $U^* = 1/U$, and thus

$$(F\sqrt{U})^* = F^*/\sqrt{U} = GU/\sqrt{U} = G\sqrt{U}.$$

For the second claim, note first that part (i) and (iii) of Lemma 2.2 imply that for $z \in \mathbb{C}^+$ we have

$$|F(z)G^*(z)| < |G(z)G^*(z)| = |F(z)F^*(z)| < |G(z)F^*(z)|.$$

Furthermore, note that $|F(\overline{z})\sqrt{U}(\overline{z})| = |F^*(z)\sqrt{U^*}(z)|$. Thus

$$\begin{aligned}|F(\overline{z})\sqrt{U}(\overline{z})|^2 &= |F^*(z)|^2|G(z)|/|F^*(z)| \\ &= |F^*(z)G(z)| > |F(z)G^*(z)| \\ &= |F(z)|^2|G^*(z)|/|F(z)| = |F(z)\sqrt{U}(z)|^2,\end{aligned}$$

and so the second claim is also established. It follows that $\mathcal{H}_0$ is a de Branges space. □

Proof of Theorem 1.1. Follows now immediately from Lemma 2.4. □

Proof of Corollary 1.2. By Theorem 1.1 there exists $b \in \mathbb{R}$ such that $e^{ibz}\mathcal{N}$ is a de Branges space. Because $\mathcal{N} \subseteq PW_a$, it follows that $e^{ibz}\mathcal{N}$ is contained in $PW_{a+|b|}$. By the de Branges ordering theorem (see [2, Theorem 35]), for every $c \in (0, a + |b|)$, either $PW_c \subseteq e^{ibz}\mathcal{N}$ or $e^{ibz}\mathcal{N} \subseteq PW_c$. Let c_0 be the supremum of c such that $PW_c \subseteq e^{ibz}\mathcal{N}$ and c_1 be the infimum of c such that $e^{ibz}\mathcal{N} \subseteq PW_c$. If $c_0 < c_1$, then for any $c \in (c_0, c_1)$ we would have that $PW_c \not\subseteq e^{ibz}\mathcal{N}$ and $e^{ibz}\mathcal{N} \not\subseteq PW_c$, which contradicts the ordering theorem. Thus $c_0 = c_1$, and consequently $PW_{c_0} = e^{ibz}\mathcal{N}$, or $e^{-ibz}PW_{c_0} = \mathcal{N}$. This shows that $\mathcal{N}$ consists precisely of the Fourier transforms of functions supported in the interval $I = (-c_0 + b, c_0 + b)$. □

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