Preface

1 Summary

The main part of my PhD thesis consists of two independent research projects. The results of the first of the projects, on certain integral operators, are contained in the following two articles.

[**Paper I**] B. Malman, *Spectra of generalized Cesàro operators acting on growth spaces*, Integral Equations and Operator Theory, 90 (2018), p. 26.

[**Paper II**] A. Limani and B. Malman, *Generalized Cesàro operators: geometry of spectra and quasi-nilpotency*, preprint (2019).

The efforts of the second project, on the backward shift operator, spawned the next two articles.

[**Paper III**] A. Aleman and B. Malman, *Density of disk algebra functions in de Branges–Rovnyak spaces*, Comptes Rendus Mathematique, 355 (2017), pp. 871–875.

[**Paper IV**] A. Aleman and B. Malman, *Hilbert spaces of analytic functions with a contractive backward shift*, arXiv:1805.11842, to appear in Journal of Functional Analysis, (2019).

Additionally, there is a short note on a topic different from the main theme of the two above projects.

[**Paper V**] B. Malman, *Nearly invariant subspaces of de Branges spaces*, arXiv:1805.11842, (2019)

Here in this introduction I will present some background material and state most important results of my thesis. For convenience, in some cases the results will be presented in a simplified form compared to what appears in the articles. I will also discuss some problems that remain unsolved and possible directions for future work.

2 Generalized Cesàro operators

2.1 Background My first research project pertains to a class of integral operators often called generalized Cesàro operators, although in the literature the alternative name Volterra-type operators also appears. The results are contained in [**Paper I**] and [**Paper II**], the second written in collaboration with my colleague Adem Limani. The operators are of the form

$$T_g f(z) = \int_0^z g'(\zeta) f(\zeta)\, d\zeta,$$

where g and f are suitable analytic functions defined in the unit disk $\mathbb{D}$. The function g is said to be the symbol of the operator T_g, and the operator is to act on some space of analytic functions which f is a member of. Some authors work with an essentially equivalent normalized version of the operator which instead acts by $f(z) \mapsto z^{-1} T_g f(z)$. The two excellent survey articles [2] and [13] explain how this class of operators appears in several problems of complex analysis and operator theory.

For a given Banach space of analytic functions X, some relevant and studied questions concerning the generalized Cesàro operators are:

(i) For what symbols g is the operator $T_g : X \to X$ bounded?

(ii) More generally, for what symbols g does the operator T_g belong to some specific class of bounded operators, e.g. compact?

(iii) If T_g is bounded, can the spectrum of T_g on X be satisfactorily characterized? How does the spectrum depend on properties of the symbol g?

Answers to the above questions have been obtained for some classical spaces of analytic functions, and references to those works are available in the introductory section of [**Paper I**]. My work in that paper is concerned with the action of T_g on the class of so-called growth spaces $A^{-\alpha}$, $\alpha > 0$, which is a family of Banach spaces each of which consists of functions $f : \mathbb{D} \to \mathbb{C}$ that satisfy

$$\|f\|_{-\alpha} := \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} |f(z)| < \infty.$$

The class appears most notably in the solutions to several famous problems of the theory of Bergman spaces. Questions (i) and (ii) above are rather easily settled for the class of growth spaces and the answers have been previously known. The boundedness and compactness of the operator T_g on $A^{-\alpha}$ is independent of $\alpha > 0$, and the corresponding conditions are that g is contained in the Bloch space $\mathcal{B}$ for boundedness, and in the little Bloch space $\mathcal{B}_0$ for compactness. The Bloch space $\mathcal{B}$ consists of functions satisfying

$$\|g\|_{\mathcal{B}} := \sup_{z \in \mathbb{D}} (1 - |z|^2) |g'(z)| < \infty,$$

while the little Bloch space $\mathcal{B}_0$ is the subspace consisting of functions satisfying

$$\lim_{r\to 1} \sup_{r<|z|<1} (1-|z|^2)|g'(z)| = 0.$$

Thus [**Paper I**] is mainly exploring the last of the questions above. The method employed originates in a very clever idea of Alexandru Aleman and Olivia Constantin from [3], where they study the analogous problem of characterizing the spectrum of T_g acting on the Bergman spaces by translating it into the task of characterizing positive weight functions $w : \mathbb{D} \to \mathbb{R}^+$ with certain properties. In the case of growth spaces, an easy argument based on their idea shows that a non-zero complex number $\lambda \in \mathbb{C}$ belongs to the complement of the spectrum of T_g acting on $A^{-\alpha}$ if and only if the weight

$$w(z) = \left|e^{\frac{g(z)}{\lambda}}\right|(1-|z|^2)^{\alpha}. \tag{1}$$

has the property that for analytic functions f we have

$$\sup_{z\in\mathbb{D}} w(z)|f(z)| \sim \sup_{z\in\mathbb{D}} w(z)(1-|z|^2)|f'(z)| + |f(0)|. \tag{2}$$

The meaning of $\sim$ here is that the two quantities are comparable, independently of f. Such a restatement of the problem facilitates the use of tools of real and complex analysis. Main part of [**Paper I**] is devoted to establishing a sufficiently useful characterization of weights w satisfying (2). Being equipped with such a weight characterization, and therefore a spectrum characterization, some interesting facts on the behaviour of the spectra of T_g operators acting on growth spaces can be derived. The rest of [**Paper I**] is devoted to this task.

Building on similar ideas and similar spectrum characterizations of T_g acting on the Hardy spaces [5] and the Bergman spaces [3], we take on the task to extend some of the results on growth spaces from [**Paper I**] to a wider range of spaces in [**Paper II**]. We also explore some completely new directions, and we find connections between the generalized Cesàro operators and certain approximation problems.

2.2 Main results The main result of [**Paper I**] characterizes the spectrum $\sigma(T_g|A^{-\alpha})$ of T_g acting on $A^{-\alpha}$. In the statement below $\rho(T_g|A^{-\alpha})$ is the resolvent set, i.e., the complement of the spectrum.

Theorem A (Theorem 5.3 in [**Paper I**]). *Assume that* $g \in \mathcal{B}$, $\lambda \in \mathbb{C}\setminus\{0\}$ *and let*

$$w(z) = \left|e^{\frac{g(z)}{\lambda}}\right|(1-|z|^2)^{\alpha}.$$

The following are equivalent:

(i) $\lambda \in \rho(T_g|A^{-\alpha})$.

(ii) For some $\delta > -1$, the weight w satisfies

$$\sup_{z\in\mathbb{D}} w(z) \int_{\mathbb{D}} \frac{1}{w(\zeta)} \frac{(1-|\zeta|^2)^\delta}{|1-z\bar{\zeta}|^{\delta+2}}\, dA(\zeta) < \infty.$$

For weights of the form (1), the condition (2) is thus equivalent to the conditions stated in the theorem. The condition in (ii) might seem a bit complicated, and it arises from an equivalent condition of boundedness of a weighted Bergman projection (see [**Paper I**] for details). It is however fully computable, and allows for explicit description of the spectrum for certain classes of symbols, for instance whenever g' is a rational function.

The characterization of Theorem *A* above can be used to derive an interesting spectral stability result. In fact, this same stability is present in Hardy and Bergman spaces, as is shown in [**Paper II**].

Theorem B (Theorem 5.4 in [**Paper I**], Theorem A in [**Paper II**]). *Let X be one the growth spaces $A^{-\alpha}$ or, for $p \in (0,\infty)$, one of the Hardy spaces H^p or Bergman spaces L^p_a. Let g, h induce bounded operators T_g and T_h on X, and further assume that $\sigma(T_h|X) = \{0\}$. Then*

$$\sigma(T_{g+h}|X) = \sigma(T_g|X).$$

The result I think is interesting in its own right, and also makes it possible to find the spectrum for a large class of symbols which arise as such perturbations of symbols with known spectra, such as symbols with rational derivative. For instance, in the case of growth spaces or the Bergman spaces perturbations by bounded analytic functions in H^∞ and functions in $\mathcal{B}_0$ do not change the spectrum, since symbols h belonging to these spaces satisfy the condition of the above theorem. In the case of Hardy spaces, the boundedness of the operator T_g is equivalent to $g \in$ **BMOA**, the space of functions of bounded mean oscillation, and it is known that $\sigma(T_h|H^p) = \{0\}$ whenever $h \in$ **VMOA**, the space of functions of vanishing mean oscillation. Our next result identifies an even larger class of symbols for which the corresponding operator spectrum consists of point 0 alone.

Theorem C (Theorem B and Corollary B of [**Paper II**]). *If g lies in the norm-closure of H^∞ in BMOA or in the norm-closure of H^∞ in $\mathcal{B}$, then we have that $\sigma(T_g|H^p) = \{0\}$ and that $\sigma(T_g|L^{p,\alpha}_a) = \sigma(T_g|A^{-\alpha}) = \{0\}$, respectively.*

Theorem *C* of course extends the applicability of Theorem *B*, but it also suggests the (perhaps naive at first sight) question of what can be said about the converse statement. If the spectrum of T_g acting on some Hardy/Bergman/growth space consists of the point zero alone, does then g lie in the norm-closure of H^∞ in **BMOA**/$\mathcal{B}$? As it turns out, this question has a positive answer in the case of Hardy spaces and in the other cases it is related to a long-standing open problem. For Hardy spaces, we established a slightly stronger result which also sheds some light on the geometric structure of the spectrum.

Theorem D (Theorem C of [**Paper II**]). *If $g \in BMOA$ is such that for some $0 < p < \infty$ the spectrum $\sigma(T_g|H^p)$ does not contain non-zero points on the real or imaginary axes, then g lies in the norm-closure of H^∞ in BMOA.*

In conjunction with Theorem *C* we easily see that in the case of Hardy spaces whenever there exist two orthogonal lines through the origin which do not intersect the spectrum in more than the point 0, then actually the spectrum consists of this point alone.

2.3 Directions for further work

The most intriguing question that has been left unanswered is if there exists a Bergman/growth space version of Theorem *D*. Unfortunately, our proof of Theorem *D* uses techniques which are very exclusive to the Hardy space setting, and it cannot be adapted to a Bergman-like setting.

Problem. *Can a function $g \in \mathcal{B}$ be approximated in the $\mathcal{B}$-norm by a bounded function, given that the spectrum of T_g on a Bergman space or a growth space equals $\{0\}$?*

The question of characterizing the closure of H^∞ in $\mathcal{B}$ has been stated in [7] and remains open to this date. Note that if the problem stated above has an affirmative solution, then a rather satisfying characterization of the closure of H^∞ in $\mathcal{B}$ would be obtained in terms of the condition (ii) of Theorem *A*, i.e., that the condition holds for the weights $w(z) = w_\lambda(z) = \left|e^{\frac{g(z)}{\lambda}}\right|(1-|z|^2)^\alpha$ for all non-zero $\lambda \in \mathbb{C}$, or in terms of the corresponding condition for the Bergman spaces (see [3] and [6], or [**Paper II**] for the exact condition characterizing the spectrum of T_g on the Bergman spaces).

3 Hilbert spaces of analytic functions with a contractive backward shift

3.1 Background

The results of the second project are contained in the papers [**Paper III**] and [**Paper IV**], both written in collaboration with my advisor Alexandru Aleman. The topic of the study is the so-called backward shift operator, denoted here by L, which acts on analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ by the formula

$$Lf(z) = \frac{f(z) - f(0)}{z}.$$

The backward shift is of major importance in operator theory for its operator modelling properties: under some very natural assumption, for any operator $T: X \to X$ on a Hilbert space X with operator norm bounded by 1, there exists a Hilbert space of analytic functions $\mathcal{H}$ such that T is unitarily equivalent to L acting on $\mathcal{H}$. In general the modelling space $\mathcal{H}$ will consist of vector-valued functions. This result is a consequence of operator model theories of de Branges-Rovnyak and of Sz.-Nagy-Foias (see [8] and [14]).

The setting for our study is a general Hilbert space $\mathcal{H}$ of analytic functions which satisfies the following properties:

(A.1) the evaluation $f \mapsto f(\lambda)$ is a bounded linear functional on $\mathcal{H}$ for each $\lambda \in \mathbb{D}$,

(A.2) $\mathcal{H}$ is invariant under the backward shift operator L and $\|Lf\|_{\mathcal{H}} \leq \|f\|_{\mathcal{H}}, f \in \mathcal{H}$,

(A.3) the constant function 1 is contained in $\mathcal{H}$ and has the reproducing property

$$\langle f, 1 \rangle_{\mathcal{H}} = f(0), f \in \mathcal{H}.$$

The class of spaces satisfying the above very general assumption is vast. Up to inessential normalization to make (A.3) hold, it includes not only the classical operator model spaces such as $K_\theta := H^2 \ominus \theta H^2$ for θ inner function, and de Branges-Rovnyak spaces $\mathcal{H}(b)$, but also classes of spaces usually not associated with model theories, such as for instance Dirichlet-type spaces.

The motivation for our research is the observation that the above three assumption imply some very non-trivial structural properties of the spaces in question, and allow for computation of the norms in the space in a very special way. This is the content of the rather technical Theorem 2.2 of [**Paper IV**], the precise statement of which I choose to omit in this exposition, and refer the reader to the paper for details. In essence however, to any space $\mathcal{H}$ in our class there exists an associated analytic row operator $\mathbf{B} := (b_i)_{i=1}^{\infty}$ such that the reproducing kernel $k_{\mathcal{H}}$ of $\mathcal{H}$ can be expressed as

$$k_{\mathcal{H}}(z, \lambda) = \frac{1 - \sum_{i \geq 1} \overline{b_i(\lambda)} b_i(z)}{1 - \overline{\lambda} z} = \frac{1 - \mathbf{B}(z)\mathbf{B}(\lambda)^*}{1 - \overline{\lambda} z} \tag{3}$$

We will denote $\mathcal{H}$ by $\mathcal{H}[\mathbf{B}]$ if the kernel is given by 3. Based on the structure of the kernel we construct a special isometric embedding operator $J : \mathcal{H} \to H^2 \oplus \Delta L^2$ of the form $Jf = (f, \mathbf{g})$, where $\Delta = \mathbf{B}^* \mathbf{B}$ and L^2 is the space of square summable vector-valued functions on the unit circle $\mathbb{T}$. The orthogonal complement of the image of this embedding is nicely characterizable in terms of $\mathbf{B}$, and certain useful intertwining relations between J and L are present. Most of the main results of the research are a derived as a consequence of the existence of this embedding.

3.2 Main results Consequences of the research include, among other results, a solution to an open problem on approximation in de Branges-Rovnyak spaces stated in [9] and a rather surprising and broad generalization, answers to questions regarding reverse Carleson embeddings from [10] together with further development of the theory of these embeddings, and the development of the theory of a natural generalization of scalar-valued de Branges-Rovnyak spaces.

3.2.1 Continuous approximation. In [9] Fricain discusses the problem of norm approximation of general functions in de Branges-Rovnyak spaces $\mathcal{H}(b)$ by functions in the disk algebra $\mathcal{A}$, the algebra of analytic functions in the disk $\mathbb{D}$ with continuous extensions to the closure of the disk clos($\mathbb{D}$). The spaces $\mathcal{H}(b)$ are precisely those Hilbert spaces of analytic functions which admit a reproducing kernel of the form given in (3) with $\mathbf{B} = (b, 0, 0, \ldots)$. The result has been known to be true in two important special cases. The first case when $\mathcal{H}(b)$ is contained isometrically inside the Hardy space H^2 is the case when b is an inner function. The density of continuous functions in this setting has been confirmed by Aleksandrov in [1]. The second case is when $\log(1 - |b|)$ is integrable with respect to the Lebesgue measure on the unit circle, that is, in the case of the so-called non-extreme de Branges-Rovnyak spaces. Sarason proved that then the analytic polynomials are contained in $\mathcal{H}(b)$, and that they moreover form a dense subset of the space.

The main result of [**Paper III**] solves in the affirmative the approximation problem for de Branges-Rovnyak spaces. The intersection $\mathcal{H}(b) \cap \mathcal{A}$ is always norm-dense in $\mathcal{H}(b)$. However, in [**Paper IV**] we have obtained a much more striking conclusion which includes the result of [**Paper III**] as a special case. The next theorem is in my view very unexpected. It is used throughout [**Paper IV**] in technical arguments, but hopefully other consequences of it will be found.

Theorem E (Theorem 3.5 in [**Paper IV**]). *Let $\mathcal{H}$ be a Hilbert space of analytic functions which satisfies the properties (A.1)-(A.3) above. Then the intersection $\mathcal{H} \cap \mathcal{A}$ is norm-dense in $\mathcal{H}$.*

In fact, a similar result holds in the case when the functions in $\mathcal{H}$ take on values in a finite dimensional Hilbert space. The function with continuous extensions to the closure of $\mathbb{D}$ are norm-dense in spaces satisfying natural versions of properties (A.1)-(A.3) in the vector-valued setting (see Section 2 of [**Paper IV**] for details).

3.2.2 Reverse Carleson measures. Theorem *E* facilitates the study of forward and reverse Carleson measures for the class of spaces considered here, and this was indeed one of the principal reasons for proving it in the special case $\mathcal{H} = \mathcal{H}(b)$ in [**Paper III**]. Our results pertain to the reverse case. A finite Borel measure on clos($\mathbb{D}$) is a reverse Carleson measure for $\mathcal{H}$ if there exists a constant $C > 0$ such that the estimate $\|f\|^2_{\mathcal{H}} \leq C \int_{\text{clos}(\mathbb{D})} |f(z)|^2 d\mu(z)$ holds at least for functions f belonging to some dense subset of $\mathcal{H}$. Note that the integral on the right-hand side does not make sense for arbitrary analytic functions f in $\mathbb{D}$, since the measure μ might contain the boundary of $\mathbb{D}$ in the support. However, for the spaces satisfying our assumptions, Theorem *E* provides us with a dense set of functions for which the integral is well-defined and on which the condition above can be tested.

We proved two theorems in the context of reverse Carleson measures. The first one should be compared to results established in [10, Theorem 2.4] which deals with the special case $\mathcal{H} = \mathcal{H}(b)$ for non-extreme b. In that case, the space $\mathcal{H}(b)$ is invariant for the operator M_z of multiplication by z: $f(z) \mapsto zf(z)$.

Theorem F (Theorem 4.2 of [**Paper IV**]). *Let $\mathcal{H}$ be a Hilbert space of analytic functions which satisfies the properties (A.1)-(A.3) above, and moreover is invariant for M_z. Then the following are equivalent.*

(i) $\mathcal{H}$ admits a reverse Carleson measure.

(ii)

$$\sup_{0<r<1} \int_{\mathbb{T}} \left\| \frac{\sqrt{1-|r\lambda|^2}}{1-r\bar{\lambda}z} \right\|_{\mathcal{H}}^2 dm(\lambda) < \infty.$$

(iii) If k is the reproducing kernel of $\mathcal{H}$, then

$$\sup_{0<r<1} \int_{\mathbb{T}} \frac{1}{(1-|r\lambda|^2)k(r\lambda, r\lambda)} dm(\lambda) < \infty.$$

If the above conditions are satisfied, then the measures $h_1 dm, h_2 dm$ on the circle given by

$$h_1(\lambda) := \lim_{r\to 1} \left\| \frac{\sqrt{1-|r\lambda|^2}}{1-r\bar{\lambda}z} \right\|_{\mathcal{H}}^2$$

and

$$h_2(\lambda) := \lim_{r\to 1} \frac{1}{(1-|r\lambda|^2)k(r\lambda, r\lambda)}$$

define reverse Carleson measures for $\mathcal{H}$. Moreover, if ν is any reverse Carleson measure for $\mathcal{H}$ and v is the density of the absolutely continuous part of the restriction of ν to $\mathbb{T}$, then $h_1 dm$ and $h_2 dm$ have the following minimality property: there exist constants $C_i > 0, i = 1, 2$ such that

$$h_i(\lambda) \le C_i v(\lambda)$$

for almost every $\lambda \in \mathbb{T}$.

We also obtain a negative result on existence of reverse Carleson measures for spaces satisfying a norm equality related to L.

Theorem G (Theorem 4.4 of [**Paper IV**]). *Let $\mathcal{H}$ be a Hilbert space of analytic functions which satisfies the properties (A.1)-(A.3) above, and moreover that the identity*

$$\|Lf\|_{\mathcal{H}}^2 = \|f\|_{\mathcal{H}}^2 - |f(0)|^2$$

holds in $\mathcal{H}$. If $\mathcal{H}$ admits a reverse Carleson measure, then $\mathcal{H}$ is isometrically contained in the Hardy space H^2.

Examples of spaces satisfying the norm condition include $\mathcal{H}(b)$ for extreme b. Consequently the above theorem answers a question in [10] whether such a space admits a reverse Carleson measure. Our result implies that this happens only in the case when b is an inner function.

3.2.3 Finite rank spaces. A generalization of $\mathcal{H}(b)$-spaces is obtained by considering reproducing kernels of the form (3) where only finitely many terms b_i are non-zero. Thus, when $\mathbf{B} = (b_1, \ldots, b_n, 0, 0 \ldots)$ and the reproducing kernel of $\mathcal{H}$ has the form

$$k_{\mathcal{H}}(z, \lambda) = \frac{1 - \sum_{i=1}^{n} \overline{b_i(\lambda)} b_i(z)}{1 - \overline{\lambda} z}.$$

In [**Paper IV**] we call such spaces finite rank spaces and we will denote them here by $\mathcal{H}[\mathbf{B}]$. We studied mainly the case when the space is invariant under the operator M_z and it turns out that under this assumption the spaces have much in common with the classical non-extreme $\mathcal{H}(b)$. For instance, for each such space there exists an n-by-n matrix-valued analytic function $\mathbf{A}$ which is a kind of Pythagorean mate of $\mathbf{B}$ in the sense that $\mathbf{B}^*\mathbf{B} + \mathbf{A}^*\mathbf{A} = I$ on the circle $\mathbb{T}$, where I is the identity matrix. An analytic function $f \in H^2$ is contained in $\mathcal{H}[\mathbf{B}]$ if and only if there exists a $\mathbb{C}^n$-valued analytic function $\mathbf{g}$ in the $\mathbb{C}^n$-valued Hardy space $H^2(\mathbb{C}^n)$ such that

$$P_+ \mathbf{B}^* f = P_+ \mathbf{A}^* \mathbf{g},$$

where P_+ is the component-wise orthogonal projection from L^2 of the circle onto H^2. If such a $\mathbf{g}$ exists, then we have that $\|f\|^2_{\mathcal{H}[\mathbf{B}]} = \|f\|^2_{H^2} + \|\mathbf{g}\|^2_{H^2}$. In fact, our isometric embedding J mentioned earlier takes f into the tuple $(f, \mathbf{g})$.

We prove some generalizations of classical theorems for $\mathcal{H}(b)$-spaces, including the density of polynomials and the structure of backward shift invariant subspaces, both with new proofs which utilize the embedding J. Moreover, we obtain the following result which sheds some light on the structure of M_z-invariant subspaces. The result is new even for $\mathcal{H}(b)$.

Theorem H (Theorem 5.11 of [**Paper IV**]). *Let $\mathcal{H} = \mathcal{H}[\mathbf{B}]$ be of finite rank, M_z-invariant and $\mathcal{M}$ be a closed M_z-invariant subspace of $\mathcal{H}$. Then*

(i) $\dim \mathcal{M} \ominus M_z\mathcal{M} = 1$,

(ii) any non-zero element in $\mathcal{M} \ominus M_z\mathcal{M}$ is a cyclic vector for $M_z|\mathcal{M}$,

(iii) if $\phi \in \mathcal{M} \ominus M_z\mathcal{M}$ is of norm 1, then there exists a space $\mathcal{H}[C]$ invariant under M_z, where $C = (c_1, \ldots, c_k)$ and $k \leq n$, such that

$$\mathcal{M} = \phi\mathcal{H}[C]$$

and the mapping $g \mapsto \phi g$ is an isometry from $\mathcal{H}[C]$ onto $\mathcal{M}$,

(iv) if $\phi \in \mathcal{M} \ominus M_z\mathcal{M}$ with $J\phi = (\phi, \boldsymbol{\phi}_1)$, then

$$\mathcal{M} = \left\{ f \in \mathcal{H}[B] : \frac{f}{\phi} \in H^2, \frac{f}{\phi}\boldsymbol{\phi}_1 \in H^2(\mathbb{C}^n) \right\}.$$

3.3 Directions for further work

The proof of Theorem E in the finite dimensional vector-valued setting is based on a duality argument which uses the characterization of the dual space of the disk algebra $\mathcal{A}$ as Cauchy transforms. For analytic functions f with continuous extensions to $\mathrm{clos}(\mathbb{D})$ which take values in an infinite dimensional vector space a similarly nice description does not seem to be available. Consequently, the following question remains open.

Problem. *Assume that $\mathcal{H}$ consists of functions taking values in an infinite dimensional vector space, and that $\mathcal{H}$ satisfies the vector-valued analogues of properties (A.1)-(A.3). Are the functions with continuous extensions to clos$(\mathbb{D})$ dense in the space?*

There is also more to be discovered about the structure of M_z-invariant subspaces. The conclusion (i) of Theorem H does not hold in the case when the hypothesis of finite rank is dropped. This follows from the work of Esterle [11], who indeed showed that $\dim \mathcal{M} \ominus M_z\mathcal{M}$ can be infinite, even when M_z is unitarily equivalent to a deceptively simple-looking weighted shift operator. The classical Dirichlet space is an example of an infinite rank space for which both the conclusion (i) and (ii) of Theorem H hold. This is a result of Stefan Richter found in [12]. Unlike the finite rank case, we have been unable to reprove his result using our model and the embedding J. It is of interest to me if these results are obtainable through these means.

An even more ambitious question is the following. Let $\mathcal{H}$ be a space of infinite rank which is M_z-invariant and in which the polynomials are dense, and let $\mathcal{M}$ be a subspace for which conclusion (i) of Theorem H is satisfied. Is then $\phi \in \mathcal{M} \ominus M_z\mathcal{M}$ a cyclic vector for M_z acting on $\mathcal{M}$? The question appears to be very difficult, and the following equivalent formulation can be deduced from our results in [**Paper IV**].

Problem. *Let $\mathcal{H}$ be a Hilbert space satisfying (A.1)-(A.3) above, is invariant for M_z and the sequence of powers of the backward shift $\{L^n\}_{n=1}^{\infty}$ converge to zero in the strong operator topology. Are the polynomials then dense in $\mathcal{H}$?*

To me personally, an explicit counterexample would be (nearly) as exciting as an affirmative solution.

4 Nearly invariant subspaces of de Branges spaces

4.1 Background

[**Paper V**] is concerned with the concept of nearly invariance. This notion appears also in the paper [**Paper IV**] in the context of function spaces on the unit disk, while here we work instead in certain spaces of entire functions called de Branges spaces.

A space of analytic functions is said to be nearly invariant if zeros of functions in the space can be divided out without leaving the space. More precisely, a space $\mathcal{H}$ is nearly invariant if for any $f \in \mathcal{H}$, the function $f(z)/(z-\lambda)$ is in $\mathcal{H}$ for any $\lambda \in \mathbb{C}$ such that

$f(\lambda) = 0$. Of course, in the case that the functions in $\mathcal{H}$ all have a zero in common at some point λ, then this zero cannot be divided out without leaving the space. Therefore, more generally, we say that a space is nearly invariant if all zeros which are not common zeros of the functions in the space can be divided out in the way indicated above.

The de Branges spaces $\mathcal{H}(E)$ form a family of Hilbert spaces of entire functions, parametrized by entire functions E which satisfy the inequality $|E(z)| > |E(\overline{z})|$ for z in the upper half plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. To define the de Branges space $\mathcal{H}(E)$ associated to E, recall the standard Hardy space of the upper half plane $H^2(\mathbb{C}^+)$, and let us define the flip operation $f^*(z) := \overline{f(\overline{z})}$, defined on entire functions. Then $\mathcal{H}(E)$ is the Hilbert space of entire functions f which satisfy the following three properties:

(i) $f/E \in H^2(\mathbb{C}^+)$,

(ii) $f^*/E \in H^2(\mathbb{C}^+)$,

(iii) $\|f\|^2_{\mathcal{H}(E)} := \int_{\mathbb{R}} |f/E|^2 dx < \infty$.

The norm of the space is the one indicated in (iii). The de Branges spaces are reproducing kernel Hilbert spaces with kernels of the form

$$k_E(\lambda, z) = \frac{E(z)\overline{E(\lambda)} - \overline{E(\overline{z})}E(\overline{\lambda})}{2\pi i(\overline{\lambda} - z)}.$$

Perhaps the most recognized Hilbert spaces of analytic functions which are de Branges spaces are the Paley-Wiener spaces PW_a, which consist of entire functions that are the Fourier transforms of measurable functions in $L^2(-a, a)$, and correspond to $E(z) = \exp(-iaz)$.

4.2 Main results

The chief theorem of [**Paper V**] is the following description of nearly invariant subspaces of de Branges spaces.

Theorem I (Theorem 1.1 in [**Paper V**]). *Let $\mathcal{N}$ be a nearly invariant subspace with no common zeros of a de Branges space $\mathcal{H}(E)$. Then there exists a de Branges space $\mathcal{H}(E_0)$ and $\alpha \in \mathbb{R}$ such that*

$$\mathcal{N} = e^{i\alpha z}\mathcal{H}(E_0) = \{e^{i\alpha z}f(z) : f \in \mathcal{H}(E_0)\}.$$

This result is used to obtain a simple proof of a more precise description from [4] in the particular case that $\mathcal{H}(E) = PW_a$. Namely, there exists an interval $I \subseteq (-a, a)$ such that

$$\mathcal{N} = \{f \in PW_a : \operatorname{supp} \hat{f} \subset I\}.$$

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