Research statement: projects and results

Bartosz Malman

December 1, 2018

1 Summary

My PhD thesis consists of two independent research projects. The results of one of the projects are contained in the published article [11], while the other project spawned the two articles [4] and [5]. Here I will present some background on each of the projects and state the main results. Some of the results will be stated in a simplified form. I will also discuss some unsolved problems and possible directions for future work.

2 Generalized Cesàro operators on growth spaces

2.1 Background My first research project pertains to a certain class of integral operators, namey the generalized Cesàro operators. The results are contained in the artice [11]. The operators are of the form

$$T_g f(z) = \int_0^z g'(\zeta) f(\zeta) \, d\zeta,$$

where g and f are suitable analytic functions defined in the unit disk $\mathbb{D}$. The function g is said to be the symbol of the operator T_g, and the operator is to act on some space of analytic functions to which f belongs. In the literature the alternative name Volterra-type operators often appears. Some authors work with an essentially equivalent normalized version which instead acts by $f(z) \mapsto z^{-1} T_g f(z)$. The two excellent survey articles [2] and [13] explain how this class of operators appears in several problems of complex analysis and operator theory.

For a given Banach space of analytic functions X, some relevant questions concerning the generalized Cesàro operators are:

(i) For what symbols g is the operator $T_g : X \to X$ bounded?

(ii) More generally, for what symbols g does the operator T_g belong to some specific class, e.g. compact?

(iii) If T_g is bounded, can the spectrum of T_g on X be satisfactorily characterized? How does the spectrum depend on properties of the symbol g?

Answers to the above questions have been obtained for many classical spaces of analytic functions, and references are available in the introductory section of my article [11].

My work was concerned with the action of T_g on the class of so-called growth spaces, which is an increasing family of Banach spaces $A^{-\alpha}, \alpha > 0$, each of which consists of functions $f : \mathbb{D} \to \mathbb{C}$ which satisfy

$$\|f\|_{-\alpha} := \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f(z)| < \infty.$$

The class appears most notably in the solutions of several famous problems of the theory of Bergman spaces. Questions (i) and (ii) above are rather easily settled for the class of growth spaces and answers have been previously known. The boundedness and compactness of the operator T_g on $A^{-\alpha}$ is independent of $\alpha > 0$, and the corresponding conditions are that g is contained in the Bloch space $\mathcal{B}$ for boundedness, and in the little Bloch space $\mathcal{B}_0$ for compactness. The Bloch space $\mathcal{B}$ consists of functions satisfying

$$\|g\|_{\mathcal{B}} := \sup_{z\in\mathbb{D}} (1 - |z|^2)|g'(z)| < \infty,$$

while the little Bloch space $\mathcal{B}_0$ is the subspace consisting of functions satisfying

$$\lim_{r\to 1} \sup_{r<|z|<1} (1 - |z|^2)|g'(z)| = 0.$$

Work in [11] is thus mainly concerned with question (iii) above. The method employed originates in a very clever idea of Alexandru Aleman and Olivia Constantin from [3] to study the problem by translating it into the task of characterizing positive weight functions $w : \mathbb{D} \to \mathbb{R}^+$ with certain properties. In the case of growth spaces, an easy argument based on their idea shows that a non-zero complex number $\lambda \in \mathbb{C}$ belongs to the spectrum of T_g acting on $A^{-\alpha}$ if and only if the weight

$$w(z) = \left|e^{\frac{g(z)}{\lambda}}\right|(1 - |z|^2)^\alpha. \tag{1}$$

has the property that for analytic functions f we have

$$\sup_{z\in\mathbb{D}} w(z)|f(z)| \sim \sup_{z\in\mathbb{D}} w(z)(1-|z|^2)|f'(z)| + |f(0)|. \tag{2}$$

Such a restatement of the problem facilitates the use of tools of real and complex analysis. Main part of [11] is devoted to establishing a sufficiently useful characterization of weights w satisfying (2).

2.2 Main results The first main result characterizes the spectrum $\sigma(T_g|A^{-\alpha})$ of T_g acting on $A^{-\alpha}$. In the statement below $\rho(T_g|A^{-\alpha})$ is the resolvent set, that is the complement of the spectrum.

Theorem A (Theorem 5.3 in [11])**.** *Assume that* $g \in \mathcal{B}, \lambda \in \mathbb{C} \setminus \{0\}$ *and*

$$w(z) = \left|e^{\frac{g(z)}{\lambda}}\right|(1-|z|^2)^\alpha.$$

The following are equivalent:

(i) $\lambda \in \rho(T_g|A^{-\alpha})$.

(ii) For some $\delta > -1$*, the weight* w *satisfies*

$$\sup_{z\in\mathbb{D}} w(z) \int_{\mathbb{D}} \frac{1}{w(\zeta)} \frac{(1-|\zeta|^2)^\delta}{|1-z\overline{\zeta}|^{\delta+2}}\, dA(\zeta) < \infty.$$

The result implies that for weights of the form (1), the condition (2) is equivalent to the conditions stated in the theorem. The condition in (ii) might seem a bit complicated, and it arises from an equivalent condition of boundedness of a weighted Bergman projection (see [11] for details). It is however fully computable, and allows for explicit description of the spectrum for certain classes of symbols, for instance whenever g' is a rational function.

The second main result sheds light on a stability property of the spectrum upon perturbations of the symbol g.

Theorem B (Theorem 5.4 in [11])**.** *Let* $g, h \in \mathcal{B}$ *and assume that* $\sigma(T_h|A^{-\alpha}) = \{0\}$*. Then*

$$\sigma(T_{g+h}|A^{-\alpha}) = \sigma(T_g|A^{-\alpha}).$$

The result I think is interesting in its own right, and also makes it possible to find the spectrum for an even larger class of symbols which arise as such perturbations of symbols with known spectra. For instance, perturbations by bounded analytic functions in H^∞ and functions in $\mathcal{B}_0$ do not change the spectrum, since symbols h belonging to these classes satisfy the condition of the above theorem. We can, for instance, prove the following theorem that the spectrum of a symbol which is the perturbation of a symbol with rational derivative is a collection of disks.

Theorem C (Theorem 5.6 in [11]). *Let* $h \in H^\infty$, $b \in \mathcal{B}_0$ *and*

$$r(z) = \sum_{k=0}^{n} c_k \log\Big(\frac{1}{1-\overline{\omega_k} z}\Big).$$

If $g = r + h + b$, *then*

$$\begin{aligned}\sigma(T_g|A^{-\alpha}) &= \bigcup_{k=1}^{n} \Big\{\lambda \in \mathbb{C} : \mathrm{Re}(c_k/\lambda) \geq \alpha\Big\} \\ &= \{0\} \cup \overline{\Big\{\lambda \in \mathbb{C}\setminus\{0\} : e^{g/\lambda} \notin A^{-\alpha}\Big\}}.\end{aligned}$$

2.3 Directions for further work Theorem B and Theorem C above can be compared to the results on spectra in [15] for the Hardy space H^2. In that paper, Scott Young shows that if g' is a rational function and b is a function of vanishing mean oscillation (which is the class inducing compact generalized Cesàro operators on H^2), then the spectra of T_g and T_{g+h} are equal. The spectrum of T_h is in that case equal to $\{0\}$, and so Young's result can be seen as a special case of Theorem B for the Hardy space.

Based on the above observation, I suggest the following direction for further study.

Problem. *Let* X *be some Banach space of analytic functions on which the operator* $T_g : X \to X$ *acts boundedly. Assume that* $T_h : X \to X$ *is also bounded and satisfies* $\sigma(T_h|X) = \{0\}$. *For what spaces* X *can we prove an analogue of Theorem* B*?*

The proof of Theorem B is based upon the characterization of the spectrum of Theorem A. Possibility of an extension of Theorem B to other classes of spaces seems to me to be most probable in the cases of weighted Bergman spaces and Hardy spaces, based on the results contained in [3, Corollary 5.2] and [6, Theorem C] which obtain characterizations of the spectrum of T_g on the respective spaces by certain conditions on certain weights, in a way somewhat similar to Theorem A.

3 Analytic function Hilbert spaces with a contractive backward shift

3.1 Background The results of the second project are contained in the papers [4] and [5], written in collaboration with my advisor Alexandru Aleman.

The topic of the study is the so-called backward shift operator, denoted here by L, which acts on analytic functions $f : \mathbb{D} \to \mathbb{C}$ by the formula

$$Lf(z) = \frac{f(z) - f(0)}{z}.$$

The backward shift is of major importance in operator theory, most prominently because of its operator modeling properties. Slightly informally, under some very natural assumption, for an operator $T : X \to X$ on a Hilbert space X with operator norm bounded by 1, there exists a Hilbert space of analytic functions $\mathcal{H}$, in general consisting of vector-valued functions, such that T is unitarily equivalent to L acting on $\mathcal{H}$. This is a consequence of operator model theories of de Branges-Rovnyak and of Sz.-Nagy-Foias (see [7] and [14]).

The setting for the study is a general Hilbert space $\mathcal{H}$ of analytic functions which satisfies the following properties.

(A.1) the evaluation $f \mapsto f(\lambda)$ is a bounded linear functional on $\mathcal{H}$ for each $\lambda \in \mathbb{D}$,

(A.2) $\mathcal{H}$ is invariant under the backward shift operator L and $\|Lf\|_{\mathcal{H}} \leq \|f\|_{\mathcal{H}}, f \in \mathcal{H}$,

(A.3) the constant function 1 is contained in $\mathcal{H}$ and has the reproducing property $\langle f, 1 \rangle_{\mathcal{H}} = f(0), f \in \mathcal{H}$.

The class of spaces satisfying the above very general assumption is vast. Up to inessential normalization to make (A.3) hold, it includes not only the classical operator model spaces such as $K_\theta := H^2 \ominus \theta H^2$ for θ inner function, and de Branges-Rovnyak spaces $\mathcal{H}(b)$, but also classes of spaces usually not associated with model theories, such as for instance Dirichlet-type spaces.

The motivation for our research is the observation that the above three assumption imply some very non-trivial structural properties of the spaces in question and allow for computation of the norms in the space in a very special way. This is the content of the rather technical Theorem 2.2 of [5], the precise statement of which I choose to omit in this exposition, and refer the reader to the article [5] for details. In essence however, to any space $\mathcal{H}$ in our class there exists a naturally associated analytic row operator $\mathbf{B} := (b_i)_{i=1}^{\infty}$ such that the reproducing kernel $k_{\mathcal{H}}$ of $\mathcal{H}$ can be expressed as

$$k_{\mathcal{H}}(z, \lambda) = \frac{1 - \sum_{i \geq 1} \overline{b_i(\lambda)} b_i(z)}{1 - \overline{\lambda} z} = \frac{1 - \mathbf{B}(z)\mathbf{B}(\lambda)^*}{1 - \overline{\lambda} z} \tag{3}$$

Based on this, we construct a special isometric embedding operator $J : \mathcal{H} \to H^2 \oplus \Delta L^2$ of the form $Jf = (f, \mathbf{g})$, where $\Delta = \mathbf{B}^*\mathbf{B}$ and L^2 is the space of square summable vector-valued functions on the unit circle $\mathbb{T}$. The orthogonal complement of the image of this embedding is nicely characterizable in terms of $\mathbf{B}$, and certain useful intertwining relations between J and L are present. Most of the main results of the research are a derived as a consequence of the existence of this embedding.

3.2 Main results Consequences of the research include, among other results, a solution to an open problem on approximation in de Branges-Rovnyak spaces stated in [8] and a rather surprising and broad generalization, answers to questions regarding reverse Carleson embeddings from [9] together with further development of the theory of these embeddings, and the development of the theory of a natural generalization of scalar-valued de Branges-Rovnyak spaces. I will discuss some samples of the these results below.

3.2.1 Continuous approximation. In [8] Fricain discusses the problem of norm-approximation of general functions in de Branges-Rovnyak spaces $\mathcal{H}(b)$ by functions in the disk algebra $\mathcal{A}$, the algebra of analytic functions in the disk $\mathbb{D}$ with continuous extensions to the closure of the disk $\text{clos}(\mathbb{D})$. The spaces $\mathcal{H}(b)$ are precisely those Hilbert spaces of analytic functions which admit a reproducing kernel of the form given in (3) with $\mathbf{B} = (b, 0, 0, \ldots)$. The result has been known to be true in two important special cases. The first case when $\mathcal{H}(b)$ is contained isometrically inside the Hardy space H^2 is the case when b is an inner function, and the density of continuous functions in this setting has been settled by Aleksandrov in [1]. The second case is when $\log(1 - |b|)$ is integrable with respect to the Lebesgue measure on the unit circle, that is, for the so-called non-extreme de Branges-Rovnyak spaces. Sarason proved that then the analytic polynomials are contained in $\mathcal{H}(b)$, and that they are moreover form a dense subset of the space.

The main result of [4] solves in the affirmative the approximation problem for de Branges-Rovnyak spaces. The intersection $\mathcal{H}(b) \cap \mathcal{A}$ is always norm-dense in $\mathcal{H}(b)$. However, in [5] we have obtained a much more striking result which includes the result of [4] as a special case.

Theorem D (Theorem 3.5 in [5]). *Let $\mathcal{H}$ be a Hilbert space of analytic functions which satisfies the properties (A.1)-(A.3) above. Then the intersection $\mathcal{H} \cap \mathcal{A}$ is norm-dense in $\mathcal{H}$.*

In fact, a similar result holds in the case when the functions in $\mathcal{H}$ take on values in a finite dimensional Hilbert space. The function with continuous extensions to the closure of $\mathbb{D}$ are norm-dense in spaces satisfying natural versions of properties (A.1)-(A.3) in the vector-valued setting. The result is in my view very unexpected. It is used throughout paper [5] in technical arguments, but hopefully many further consequences of it will be found.

3.2.2 Reverse Carleson measures. Theorem D facilitates the study of forward and reverse Carleson measures for the class of spaces considered here, and this was indeed one of the principal reasons for proving the special case of $\mathcal{H} = \mathcal{H}(b)$ in [4]. Our results pertain to the reverse case. A finite Borel measure on clos($\mathbb{D}$) is a reverse Carleson measure for $\mathcal{H}$ if there exists a constant $C > 0$ such that the estimate $\|f\|_{\mathcal{H}}^2 \leq C \int_{\text{clos}(\mathbb{D})} |f(z)|^2 d\mu(z)$ holds at least for functions f belonging to some dense subset of $\mathcal{H}$. Note that the integral on the right-hand side does not make sense for arbitrary analytic functions f in $\mathbb{D}$, since the measure μ might contain the boundary of $\mathbb{D}$ in the support. However, for the spaces satisfying our assumptions, Theorem D provides us with a dense set of functions for which the integral is well-defined and on which the condition above can be tested.

We proved two theorems in the context of reverse Carleson measures. The first one should be compared to results established in [9, Theorem 2.4] which deals with the special case $\mathcal{H} = \mathcal{H}(b)$ for non-extreme b. In that case, the space $\mathcal{H}(b)$ is invariant for the operator M_z of multiplication by z: $f(z) \mapsto zf(z)$.

Theorem E (Theorem 4.2 of [5]). *Let $\mathcal{H}$ be a Hilbert space of analytic functions which satisfies the properties (A.1)-(A.3) above, and moreover is invariant for M_z. Then the following are equivalent.*

(i) $\mathcal{H}$ admits a reverse Carleson measure.

(ii)

$$\sup_{0<r<1} \int_{\mathbb{T}} \left\| \frac{\sqrt{1-|r\lambda|^2}}{1-r\overline{\lambda}z} \right\|_{\mathcal{H}}^2 dm(\lambda) < \infty.$$

(iii) If k is the reproducing kernel of $\mathcal{H}$, then

$$\sup_{0<r<1} \int_{\mathbb{T}} \frac{1}{(1-|r\lambda|^2)k(r\lambda, r\lambda)} dm(\lambda) < \infty.$$

If the above conditions are satisfied, then the measures $h_1 dm, h_2 dm$ on the circle given by

$$h_1(\lambda) := \lim_{r\to 1} \left\| \frac{\sqrt{1-|r\lambda|^2}}{1-r\overline{\lambda}z} \right\|_{\mathcal{H}}^2$$

and

$$h_2(\lambda) := \lim_{r\to 1} \frac{1}{(1-|r\lambda|^2)k(r\lambda, r\lambda)}$$

define reverse Carleson measures for $\mathcal{H}$. Moreover, if ν is any reverse Carleson measure for $\mathcal{H}$ and v is the density of the absolutely continuous part of the restriction of ν to $\mathbb{T}$, then $h_1 dm$ and $h_2 dm$ have the following minimality property: there exist constants $C_i > 0, i = 1, 2$ such that

$$h_i(\lambda) \leq C_i v(\lambda)$$

for almost every $\lambda \in \mathbb{T}$.

We also obtain a negative result on existence of reverse Carleson measures for spaces satisfying certain norm equality related to L.

Theorem F (Theorem 4.4 of [5])**.** *Let $\mathcal{H}$ be a Hilbert space of analytic functions which satisfies the properties (A.1)-(A.3) above, and moreover that the identity*

$$\|Lf\|_{\mathcal{H}}^2 = \|f\|_{\mathcal{H}}^2 - |f(0)|^2$$

holds in $\mathcal{H}$. If $\mathcal{H}$ admits a reverse Carleson measure, then $\mathcal{H}$ is isometrically contained in the Hardy space H^2.

Examples of spaces satisfying the norm condition include $\mathcal{H}(b)$ for extreme b. Consequently the above theorem answers a question in [9] whether such a space admits a reverse Carleson measure. Our result implies that this happens only in the case when b is an inner function.

3.2.3 Finite rank spaces. A generalization of $\mathcal{H}(b)$-spaces is obtained by considering reproducing kernels of the form (3) where finitely many terms b_i are non-zero, thus when $\mathbf{B} = (b_1, \ldots, b_n, 0, 0 \ldots)$ and the reproducing kernel of $\mathcal{H}$ has the form

$$k_{\mathcal{H}}(z, \lambda) = \frac{1 - \sum_{i=1}^{n} \overline{b_i(\lambda)} b_i(z)}{1 - \overline{\lambda} z}.$$

In [5] we call such spaces finite rank spaces and we will denote them here by $\mathcal{H}[\mathbf{B}]$. We studied mainly the case when the space is invariant under the operator M_z and it turns out that under this assumption the spaces have much in common with the classical non-extreme $\mathcal{H}(b)$. For instance, to each such space there corresponds an n-by-n matrix-valued analytic function $\mathbf{A}$ which is a kind of Pythagorean mate of $\mathbf{B}$ in the sense that $\mathbf{B}^*\mathbf{B} + \mathbf{A}^*\mathbf{A} = I$ on the circle $\mathbb{T}$, where I is the identity matrix. An analytic function $f \in H^2$

is contained in $\mathcal{H}[\mathbf{B}]$ if and only if there exists a $\mathbb{C}^n$-valued analytic function $\mathbf{g}$ in the $\mathbb{C}^n$-valued Hardy space $H^2(\mathbb{C}^n)$ such that

$$P_+\mathbf{B}^*f = P_+\mathbf{A}^*\mathbf{g},$$

where P_+ is the component-wise orthogonal projection from L^2 of the circle onto H^2. If such a $\mathbf{g}$ exists, then we have that $\|f\|^2_{\mathcal{H}[\mathbf{B}]} = \|f\|^2_{H^2} + \|\mathbf{g}\|^2_{H^2}$. In fact, our isometric embedding J mentioned earlier takes f into the tuple $(f, \mathbf{g})$.

We prove some generalizations of classical theorems for $\mathcal{H}(b)$-spaces, including the density of polynomials and the structure of backward shift invariant subspaces, both with new proofs which utilize the embedding J. Moreover, we obtain the following result which sheds some light on the structure of M_z-invariant subspaces. The result is new even for $\mathcal{H}(b)$.

Theorem G (Theorem 5.11 of [5]). *Let $\mathcal{H} = \mathcal{H}[\boldsymbol{B}]$ be of finite rank and M_z-invariant and $\mathcal{M}$ be a closed M_z-invariant subspace of $\mathcal{H}$. Then*

(i) $\dim \mathcal{M} \ominus M_z\mathcal{M} = 1$,

(ii) any non-zero element in $\mathcal{M} \ominus M_z\mathcal{M}$ is a cyclic vector for $M_z|\mathcal{M}$,

(iii) if $\phi \in \mathcal{M} \ominus M_z\mathcal{M}$ is of norm 1, then there exists a space $\mathcal{H}[\boldsymbol{C}]$ invariant under M_z, where $\boldsymbol{C} = (c_1, \ldots, c_k)$ and $k \leq n$, such that

$$\mathcal{M} = \phi\mathcal{H}[\boldsymbol{C}]$$

and the mapping $g \mapsto \phi g$ is an isometry from $\mathcal{H}[\boldsymbol{C}]$ onto $\mathcal{M}$,

(iv) if $\phi \in \mathcal{M} \ominus M_z\mathcal{M}$ with $J\phi = (\phi, \boldsymbol{\phi}_1)$, then

$$\mathcal{M} = \left\{ f \in \mathcal{H}[\boldsymbol{B}] : \tfrac{f}{\phi} \in H^2, \tfrac{f}{\phi}\boldsymbol{\phi}_1 \in H^2(\mathbb{C}^n) \right\}.$$

3.3 Directions for further work The proof of Theorem D in the finite dimensional vector-valued setting is based on a duality argument which uses the characterization of the dual space of the disk algebra $\mathcal{A}$ as Cauchy transforms. For analytic functions f with continuous extensions to clos($\mathbb{D}$) which take values in an infinite dimensional vector space a similarly nice description does not seem to be possible. In particular, the following question remains open.

Problem. *Assume that $\mathcal{H}$ consists of functions taking values in an infinite dimensional vector space, and that $\mathcal{H}$ satisfies the vector-valued analogues of properties (A.1)-(A.3) (see [5, Section 2] for details). Are the functions with continuous extensions to clos($\mathbb{D}$) dense in the space?*

There is also more to be discovered about the structure of M_z-invariant subspaces. The conclusion (i) of Theorem G does not hold in the case when the hypothesis of finite rank is dropped. This follows from the work of Esterle [10], who indeed showed that $\dim \mathcal{M} \ominus M_z \mathcal{M}$ can be infinite, even when M_z is unitarily equivalent to a deceptively simple-looking weighted shift operator. The classical Dirichlet space is an example of an infinite rank space for which both the conclusion (i) and (ii) of Theorem G hold. This is a result of Stefan Richter found in [12]. Unlike the finite rank case, we have been unable to reprove his result using our model and the embedding J. It is of interest to me if these results are obtainable through these means.

An even more ambitious question is the following. Let $\mathcal{H}$ be a space of infinite rank which is M_z-invariant, and let $\mathcal{M}$ be a subspace for which conclusion (i) of Theorem G is satisfied. Is then $\phi \in \mathcal{M} \ominus M_z\mathcal{M}$ a cyclic vector for M_z acting on $\mathcal{M}$? The question appears to be very difficult, and the following equivalent formulation can be deduced from our results in [5].

Problem. *Let $\mathcal{H}$ be a Hilbert space satisfying (A.1)-(A.3) above, is invariant for M_z and the sequence of powers of the backward shift $\{L^n\}_{n=1}^{\infty}$ converge to zero in the strong operator topology. Are the polynomials then dense in $\mathcal{H}$?*

To me personally, an explicit counterexample would be (nearly) as exciting as an affirmative solution.

References

[1] A. B. Aleksandrov, *Invariant subspaces of shift operators. An axiomatic approach*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 113 (1981), pp. 7–26, 264.

[2] A. Aleman, *A class of integral operators on spaces of analytic functions*, in Topics in complex analysis and operator theory, Univ. Málaga, Málaga, 2007, pp. 3–30.

[3] A. Aleman and O. Constantin, *Spectra of integration operators on weighted Bergman spaces*, J. Anal. Math., 109 (2009), pp. 199–231.

[4] A. Aleman and B. Malman, *Density of disk algebra functions in de Branges–Rovnyak spaces*, Comptes Rendus Mathematique, 355 (2017), pp. 871–875.

[5] ——, *Hilbert spaces of analytic functions with a contractive backward shift*, arXiv preprint arXiv:1805.11842, to appear in Journal of Functional Analysis, (2018).

[6] A. Aleman and J. A. Peláez, *Spectra of integration operators and weighted square functions*, Indiana Univ. Math. J., 61 (2012), pp. 775–793.

[7] J. Ball and V. Bolotnikov, *de Branges-Rovnyak spaces: basics and theory*, pp. 631–679.

[8] C. Bénéteau, A. Condori, C. Liaw, W. Ross, and A. Sola, *Some open problems in complex and harmonic analysis: report on problem session held during the conference Completeness problems, Carleson measures, and spaces of analytic functions*, in Recent progress on operator theory and approximation in spaces of analytic functions, vol. 679 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2016, pp. 207–217.

[9] A. Blandignères, E. Fricain, F. Gaunard, A. Hartmann, and W. Ross, *Direct and reverse Carleson measures for $H(b)$ spaces*, Indiana Univ. Math. J., 64 (2015), pp. 1027–1057.

[10] J. Esterle, *Toeplitz operators on weighted Hardy spaces*, Algebra i Analiz, 14 (2002), pp. 92–116.

[11] B. Malman, *Spectra of generalized Cesàro operators acting on growth spaces*, Integral Equations and Operator Theory, 90 (2018), p. 26.

[12] S. Richter, *Invariant subspaces of the Dirichlet shift*, J. Reine Angew. Math., 386 (1988), pp. 205–220.

[13] A. G. Siskakis, *Volterra operators on spaces of analytic functions—a survey*, in Proceedings of the First Advanced Course in Operator Theory and Complex Analysis, Univ. Sevilla Secr. Publ., Seville, 2006, pp. 51–68.

[14] B. Sz.-Nagy, C. Foias, H. Bercovici, and L. Kérchy, *Harmonic analysis of operators on Hilbert space*, Universitext, Springer, New York, second ed., 2010.

[15] S. W. Young, *Spectral properties of generalized Cesàro operators*, Integral Equations Operator Theory, 50 (2004), pp. 129–146.