REGULAR FUNCTIONS IN DE BRANGES-ROVNYAK SPACES

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Abstract. In this expository article, we discuss the problem of the containment of a non-zero function satisfying various boundary regularity conditions in the de Branges-Rovnyak space $\mathcal{H}(b)$. We review what is known, and highlight the connection of this problem to the Korenblum-Roberts cyclicity theory in Bergman-type spaces, the works of Khrushchev on simultaneous approximation by polynomials, and some aspects of the Beurling-Malliavin theory. New proofs of previously known results are given to emphasize these connections. As a new contribution, we fully characterize the symbols b for which the space $\mathcal{H}(b)$ contains a non-zero function with C^∞ extension to the boundary. This extends an earlier result of Dyakonov and Khavinson dealing with the case of inner b. We end the article by stating a few open problems.

1. Introduction

1.1. **Smoothness classes.** Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk of the complex plane $\mathbb{C}$. If an analytic function $f : \mathbb{D} \to \mathbb{C}$ extends continuously to the boundary $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, then we say that f is a member of the *disk algebra* $\mathcal{A}$. In this article, we will study functions which admit an extension to $\mathbb{T}$ with additional regularity properties. We ask if a non-zero function with such properties can be found in a de Branges-Rovnyak space $\mathcal{H}(b)$, a classical family of spaces of analytic functions in $\mathbb{D}$ parametrized by analytic selfmaps $b : \mathbb{D} \to \mathbb{D}$.

We will restrict our attention to a few often appearing regularity classes.

(1) For an integer $n \geq 1$, we denote by $\mathcal{A}^n$ the class of functions in $\mathbb{D}$ for which the n:th derivative $f^{(n)}$ is contained in $\mathcal{A}$. We use the conventions $\mathcal{A}^0 := \mathcal{A}$ and $\mathcal{A}^\infty := \cap_{n \geq 0} \mathcal{A}^n = \mathcal{A} \cap C^\infty(\mathbb{T})$.

(2) For $\alpha \in (0, 1]$, the analytic Hölder class Λ^α_a consists of those functions $f \in \mathcal{A}$ which satisfy the modulus of continuity estimate

$$|f(z) - f(w)| \leq C_f |z - w|^\alpha, \quad z, w \in \mathbb{D} \cup \mathbb{T},$$

for some constant $C_f > 0$.

(3) The analytic Gevrey class $\mathcal{G}_\beta$, $\beta \in (0, 1]$, consists of those $f \in \mathcal{A}$ whose Fourier coefficients $\widehat{f}(n)$, $n \geq 0$, decay rapidly in the sense that

$$|\widehat{f}(n)| \leq C_f \exp(-D_f n^\beta)$$

for some $C_f, D_f > 0$. We have $\mathcal{G}_\beta \subset \mathcal{A}^\infty$ for every β, but the Gevrey classes are much smaller than $\mathcal{A}^\infty$. Note that the class $\mathcal{G}_1$ consists precisely of those functions f in $\mathbb{D}$ which have a holomorphic extension to a disk containing $\mathbb{D} \cup \mathbb{T}$.

To a reader unfamiliar with the spaces $\mathcal{H}(b)$, it might come as a surprise that the question of existence of a regular function in an $\mathcal{H}(b)$-space doesn't have a trivial answer. After all, most spaces of analytic functions in $\mathbb{D}$ appearing in function and operator theory, such as Hardy, Bergman and Dirichlet-type spaces, are readily seen to contain all functions with a sufficiently smooth extension to $\mathbb{T}$. The special case of $\mathcal{H}(b)$-spaces is much different, and one of the purposes of this article is to explain how this problem connects to some deep results of 20th century analysis: Korenblum's work on cyclic singular inner functions, Khrushchev's results on simultaneous approximation by polynomials, and even some aspects of the Beurling-Malliavin theory. These connections will be made explicit in Section 3 below, which constitutes the main part of this article. Certain results that we present are new, but for the most part the content of the article deals with previously established theory. The focus is on presenting a unified account of the containment problem, and its relationship to other parts of operator theory and complex analysis. For this reason, we often provide new proofs of previously known results.

A non-zero function analytic in a disk larger than $\mathbb{D}$ (that is, a function in $\mathcal{G}_1$) is contained in a given space $\mathcal{H}(b)$ if and only if b satisfies some very particular and readily verified conditions. More precisely, we must have either that $b(\lambda) = 0$ for some $\lambda \in \mathbb{D}$, or that

$$\int_{\mathbb{T}} \log(1 - |b|^2)\, dm > -\infty. \tag{1.1}$$

Here dm is the Lebesgue measure on $\mathbb{T}$, and b is defined on $\mathbb{T}$ in the sense of its non-tangential boundary values (which exist almost everywhere with respect to dm). This result is well-known and essentially contained in the standard reference books by Sarason [34] and Fricain-Mashreghi [14], [15]. Condition (1.1) is known to characterize the symbols b for which the analytic polynomials are contained and dense in the space $\mathcal{H}(b)$, and is known to be equivalent to b being a non-extreme point of the unit ball of $\mathcal{H}^\infty$, the algebra of bounded analytic functions in $\mathbb{D}$. We will therefore be concerned with the case when the left-hand side integral in (1.1) diverges, and so b and the corresponding space $\mathcal{H}(b)$ are *extreme*. The goal is to convince the reader that the convergence of the integral (1.1) over sets smaller than $\mathbb{T}$ is decisive for existence of non-zero functions of a given regularity in the space $\mathcal{H}(b)$.

1.2. **Construction of the space.** For $p \in (0, \infty)$, the Hardy space $\mathcal{H}^p$ is defined, as usual, to be the space of analytic functions satisfying the uniform integral mean bound

$$\sup_{r \in (0,1)} \int_{\mathbb{T}} |f(r\zeta)|^p dm(\zeta) < \infty.$$

The earlier mentioned algebra $\mathcal{H}^\infty$ consists of analytic functions which are bounded in $\mathbb{D}$. It is well known that each function $f \in \mathcal{H}^p$ admits a non-tangential boundary value $f(\zeta)$ for almost every $\zeta \in \mathbb{T}$ with respect to dm, and that we have $\int_{\mathbb{T}} |f(\zeta)|^p\, dm < \infty$. Through the identification of the function $f \in \mathcal{H}^p$ with its boundary function on $\mathbb{T}$, we may consider $\mathcal{H}^p$ as a subspace of $L^p(dm)$. For $p \in [1, \infty]$, we have also the alternative description in terms of Fourier coefficients:

$$\mathcal{H}^p := \big\{ f \in L^p(dm) : \widehat{f}(n) = 0, n < 0 \big\}. \tag{1.2}$$

That is, $\mathcal{H}^p$ is the closed subspace of $L^p(dm)$ consisting of functions with vanishing negative Fourier coefficients $\widehat{f}(n)$.

Given analytic $b : \mathbb{D} \to \mathbb{D}$, we have the *Nevanlinna factorization* (see [17, Chapter II] for details):

$$b = cBS_\nu b_0, \tag{1.3}$$

where c is a unimodular constant (for convenience we will assume $c = 1$), B is the usual Blaschke product corresponding to the zero set $\{\lambda_k\}_k$ of b in $\mathbb{D}$

$$B(z) = \prod_k \frac{|\lambda_k|}{\lambda_k} \frac{\lambda_k - z}{1 - \overline{\lambda_k} z}, \quad z \in \mathbb{D},$$

S_ν is a singular inner function corresponding to a finite singular non-negative Borel measure ν on $\mathbb{T}$:

$$S_\nu(z) = \exp\Big(- \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\nu(\zeta) \Big), \quad z \in \mathbb{D}, \tag{1.4}$$

and b_0 is the outer factor

$$b_0(z) = \exp\Big(- \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log|b(\zeta)| \Big), \quad z \in \mathbb{D}.$$

To avoid trivialities, we will always assume that b is non-constant.

The space $\mathcal{H}(b)$ can be defined as the Hilbert space of analytic functions on $\mathbb{D}$ with the following reproducing kernel:

$$k_b(\lambda, z) := \frac{1 - \overline{b(\lambda)} b(z)}{1 - \overline{\lambda} z} = (1 - \overline{b(\lambda)} b(z)) k_\lambda(z). \tag{1.5}$$

Here $k_\lambda(z) := (1 - \overline{\lambda} z)^{-1}$ is the Szegö kernel. More precisely, we require that

$$\big\langle f, k_b(\lambda, \cdot) \big\rangle_b = f(\lambda), \quad \lambda \in \mathbb{D}. \tag{1.6}$$

where $\langle \cdot, \cdot \rangle_b$ denotes the inner product of $\mathcal{H}(b)$. This way of defining the space doesn't really give us any clues as to what functions may be contained in $\mathcal{H}(b)$, beside $k_b(\lambda, \cdot)$. In the special case that $b = BS_\nu = \theta$ is an inner function, the space $\mathcal{H}(b)$ is known to coincide isometrically with the so-called *model space* K_θ which is the orthogonal complement of the subspace $\theta\mathcal{H}^2 = \{\theta h : h \in \mathcal{H}^2\}$ in $\mathcal{H}^2$:

$$K_\theta := \{f \in \mathcal{H}^2 : \langle f, \theta h \rangle = 0, h \in \mathcal{H}^2\}. \tag{1.7}$$

Above $\langle \cdot, \cdot \rangle$ denotes the inner product of $L^2(dm)$. For more background regarding model spaces, see [16].

In the general case, we can construct the space $\mathcal{H}(b)$ as the image of the operator

$$\sqrt{I - \mathcal{T}_b \mathcal{T}_{\overline{b}}} : \mathcal{H}^2 \to \mathcal{H}^2. \tag{1.8}$$

Here I denotes the identity operator on $\mathcal{H}^2$, and $\mathcal{T}_b$, $\mathcal{T}_{\overline{b}}$ are Toeplitz operators (see Section 2.1). The operator in (1.8) is injective unless $b = \theta$ is an inner function, in which case $\sqrt{I - \mathcal{T}_\theta \mathcal{T}_{\overline{\theta}}}$ is simply the orthogonal projection from $\mathcal{H}^2$ onto the space K_θ in (1.7). If b is not an inner function, then to each $f \in \mathcal{H}(b)$ there corresponds a unique $g \in \mathcal{H}^2$ satisfying $f = \sqrt{I - \mathcal{T}_b \mathcal{T}_{\overline{b}}}\, g$, and for $f_i = \sqrt{I - \mathcal{T}_b \mathcal{T}_{\overline{b}}} g_i$, $i = 1, 2$, we have

$$\langle f_1, f_2 \rangle_b := \langle g_1, g_2 \rangle.$$

A short computation reveals that

$$k_b(\lambda, \cdot) = (I - \mathcal{T}_b \mathcal{T}_{\overline{b}}) k_\lambda \in \mathcal{H}(b),$$

with k_b as in (1.5), is indeed the reproducing kernel of $\mathcal{H}(b)$. We note that the space $\mathcal{H}(b)$ is always contained in $\mathcal{H}^2$.

1.3. **Where we are going.** The two objects which contain the information we are trying to extract are the singular measure ν in (1.3), and the measurable non-negative function

$$\Delta_b(\zeta) := \sqrt{1 - |b(\zeta)|^2}, \quad \zeta \in \mathbb{T}. \tag{1.9}$$

The following is a brief outline of our strategy, in which the weight (1.9) plays the leading role. It is known that if $f \in \mathcal{H}(b)$, then $\mathcal{T}_{\overline{b}} f$ is contained in $\mathcal{H}(b)$ also, and can be represented as $P_+ k\Delta_b$, an orthogonal projection to $\mathcal{H}^2$ of the function $k\Delta_b \in L^2(dm)$, where $k \in L^2(dm)$ (see Proposition 2.1 below). Generally speaking, if f satisfies a regularity condition on $\mathbb{T}$, then $\mathcal{T}_{\overline{b}} f$ satisfies a similar regularity condition (see Proposition 2.3 and Remark 2.5 below). In this way, we see that regular functions in $\mathcal{H}(b)$ correspond to functions on $\mathbb{T}$ of the form $k\Delta_b$ which have regular projections. Extensions of Khrushchev's approximation theory from [22] and simple special cases of the Beurling-Malliavin theorem from [8] can be used to study these projections. In the transformation $f \mapsto \mathcal{T}_{\overline{b}} f$ we lose some information: the operator $\mathcal{T}_{\overline{b}}$ has a kernel which equals K_{θ_b}, θ_b being the inner factor of b. To study regular functions in the kernel, we use the Korenblum-Roberts theory of inner functions from [23], [24], [25] and [33].

For the most part we provide complete proofs. At some points, we refer to prior works for details of the less important parts. At the end of our discussion, we will have presented a proof of the following theorem. The definition of a Carleson set, appearing in the following statement, is given in (3.4) below.

Theorem. *Let $b = BS_\nu b_0$ be the Nevanlinna factorization of b.*

- *(A) The space $\mathcal{H}(b)$ always contains a non-zero function in the disk algebra $\mathcal{A}$.*
- *(B) The space $\mathcal{H}(b)$ contains a non-zero function in the Hölder class Λ^α_a, $\alpha \in (0, 1]$, if and only if it contains a non-zero function in $\mathcal{A}^\infty$. This occurs if and only if at least one of the following three conditions is satisfied:*
  - *(i) there exists $\lambda \in \mathbb{D}$ for which $b(\lambda) = 0$,*
  - *(ii) there exists a Carleson set E of zero Lebesgue measure for which $\nu(E) > 0$,*
  - *(iii) there exists a Carleson set E of positive Lebesgue measure for which $\int_E \log \Delta_b \, dm > -\infty$.*
- *(C) The space $\mathcal{H}(b)$ contains a non-zero function in the Gevrey class $\mathcal{G}_\beta$ for $\beta \in [1/2, 1)$ if and only if at least one of the following two conditions is satisfied:*
  - *(i) there exists $\lambda \in \mathbb{D}$ for which $b(\lambda) = 0$,*
  - *(ii) there exists an arc I of $\mathbb{T}$ for which $\int_I \log \Delta_b \, dm > -\infty$.*
- *(D) The space $\mathcal{H}(b)$ contains a non-zero function which extends analytically to a disk larger than $\mathbb{D}$ if and only if at least one of the following two conditions is satisfied:*

(i) *there exists* $\lambda \in \mathbb{D}$ *for which* $b(\lambda) = 0$,
(ii) *the following global integrability condition holds:* $\int_{\mathbb{T}} \log \Delta_b \, dm > -\infty$.

Except for part (B), which is an extension of a result of Dyakonov and Khavinson from [12], the statements are not new. Part (B) follows from Theorem 3.8 and Remark 3.19 below. Part (A) is known in stronger form as a universal density theorem of $\mathcal{H}(b) \cap \mathcal{A}$ in $\mathcal{H}(b)$ from [3]. We will give a new proof of statement (A) which relies on Khrushchev's theory from [22] (see Corollary 3.7 below). Part (C) is from [32], and we give a condensed version of the proof (Theorem 3.22 below). Part (D) has already been mentioned above, it is a well-known result of the theory of $\mathcal{H}(b)$-spaces.

Evidently, the Gevrey classes $\mathcal{G}_\beta$ corresponding to the parameter range $\beta \in (0, 1/2)$ are missing in the above description. In this case, a statement similar to part (C) above is very likely to hold. More on this will be said in Section 4, where we state a related conjecture. The reader will hopefully be inspired to fill in this annoying gap.

2. Preliminaries

We will need only very few background results on $\mathcal{H}(b)$-spaces, and mainly those which pertain to the action of coanalytic Toeplitz operators. We start this section by presenting those results. Then, we introduce a few useful Hilbert spaces, and discuss mapping properties of coanalytic Toeplitz operators on them. We end the section by a brief reminder of Cauchy duality between spaces of analytic functions in $\mathbb{D}$.

2.1. **Projections and Toeplitz operators.** Throughout the article, the operator

$$P_+ : \mathcal{L}^2(dm) \to \mathcal{H}^2$$

will denote the orthogonal projection. In terms of Fourier series

$$g = \sum_{n \in \mathbb{Z}} \widehat{g}(n) \zeta^n, \quad \zeta \in \mathbb{T}$$

we have

$$P_+ g = \sum_{n \geq 0} \widehat{g}(n) \zeta^n.$$

The Fourier series representation shows that if $g \in C^\infty(\mathbb{T})$, then $P_+ g \in \mathcal{A}^\infty = \mathcal{A} \cap C^\infty(\mathbb{T})$. Indeed, $g \in C^\infty(\mathbb{T})$ if and only if we have the spectral decay $|\widehat{g}(n)| \leq C_A |n|^{-A}$ for every $A > 0$, where $C_A > 0$ is some constant.

For a bounded function g on $\mathbb{T}$, the Toeplitz operator $\mathcal{T}_g : \mathcal{H}^2 \to \mathcal{H}^2$ is defined as

$$\mathcal{T}_g : f \mapsto P_+ g f.$$

We shall exclusively be interested in operators with coanalytic symbols $g = \overline{h}$, $h \in \mathcal{H}^\infty$. The space $\mathcal{H}(b)$ is invariant under the coanalytic Toeplitz operators (see [15, Theorem 18.13]). In particular, it is invariant under the *backward shift operator* $L := \mathcal{T}_{\overline{z}}$,

$$Lf(z) = \frac{f(z) - f(0)}{z}. \tag{2.1}$$

The kernel of the coanalytic Toeplitz operator $\mathcal{T}_{\overline{h}}$ equals the model space K_{θ_h} defined in (1.7), where θ_h is the inner factor of h. Thus $\mathcal{T}_{\overline{h}}$ is injective if and only if h is outer.

An important operator corresponds to the symbol b. It has special properties.

Proposition 2.1. *Let* b_0 *be the outer factor of* b*. The operator* $\mathcal{T}_{\overline{b}}$ *maps* $\mathcal{H}(b)$ *into* $\mathcal{H}(b_0)$*, and moreover, for each* $f \in \mathcal{H}(b)$ *we have that*

$$\mathcal{T}_{\overline{b}} f = P_+ \Delta_b k$$

for some $k \in L^2(dm)$*, where* Δ_b *is given by* (1.9).

The statement is well-known to specialists of the $\mathcal{H}(b)$-theory, and follows from a computation and an application of *Douglas' criterion* from [11] on operator range containment. For a proof see, for instance, [15, Theorem 17.8 and Corollary 25.2]. In fact, all projections of the type $P_+ \Delta_b k$ are contained in $\mathcal{H}(b)$.

Proposition 2.2. *For any* $k \in L^2(dm)$*, the function* $P_+ \Delta_b k$ *is contained in* $\mathcal{H}(b)$*.*

Proof. By [15, Theorem 17.9 and Theorem 20.1], if k_0 is a function on $\mathbb{T}$ which satisfies

$$\int_{\mathbb{T}} |k_0|^2 \Delta_b^2 dm < \infty, \tag{2.2}$$

then $P_+\Delta_b^2 k_0 \in \mathcal{H}(b)$. Without loss of generality we may assume that that $k \in L^2(dm)$ lives only on the set $\{\zeta \in \mathbb{T} : \Delta_b(\zeta) > 0\}$, this set being well-defined up to a set of dm-measure zero. Then clearly $k_0 := k/\Delta_b$ satisfies (2.2), and so $P_+\Delta_b^2 k_0 = P_+\Delta_b k \in \mathcal{H}(b)$. $\square$

2.2. **Useful Toeplitz-invariant Hilbert spaces.** Our study will be focused on the regularity classes Λ_a^α, $\mathcal{A}^\infty$ and $\mathcal{G}_\beta$. Topologies on these spaces may be defined, which are however quite complicated from the point of view of functional analysis. So are their duals. It will be convenient to replace these spaces with related ones which are topologically simpler. We will see that the containment problem studied here is insensitive to this replacement.

With the above remark in mind, we introduce the Hilbert spaces $\mathcal{H}_\alpha^2$, $\alpha \in \mathbb{R}$, which we define as

$$\mathcal{H}_\alpha^2 := \Big\{ f(z) = \sum_{n\geq 0} f_n z^n : \|f\|_{\mathcal{H}_\alpha^2}^2 := \sum_{n \geq 0} (n+1)^\alpha |f_n|^2 < \infty \Big\}, \tag{2.3}$$

and the Gevrey-type Hilbert spaces $\mathcal{GH}_{\beta,c}^2$, $\beta \in (0,1)$, $c \in \mathbb{R}$, which are

$$\mathcal{GH}_{\beta,c}^2 := \Big\{ f(z) = \sum_{n\geq 0} f_n z^n : \|f\|_{\mathcal{GH}_{\beta,c}^2}^2 := \sum_{n \geq 0} \exp(cn^\beta) |f_n|^2 < \infty \Big\} \tag{2.4}$$

The following is the smoothness preservation property of coanalytic Toeplitz operators which we will exploit.

Proposition 2.3. *If $\alpha \geq 0$, then $\mathcal{H}_\alpha^2$ is invariant for the coanalytic Toeplitz operators. If $c \geq 0$, then $\mathcal{GH}_{\beta,c}^2$ is invariant for the coanalytic Toeplitz operators.*

Sketch of proof. Note that the backward shift operator $L = \mathcal{T}_{\overline{z}}$ is a contraction on the Hilbert space $\mathcal{H}_\alpha^2$ whenever $\alpha \geq 0$, and on $\mathcal{GH}_{\beta,c}^2$ whenever $c \geq 0$. Therefore, in both cases, to each $h \in \mathcal{H}^\infty$ there corresponds an operator $h(L)$ acting on the space, defined through the Sz. Nagy-Foias functional calculus (see [35, Chapter III]). For a polynomial p, we see readily that $p(L)$ is the coanalytic Toeplitz operator with symbol $p(\overline{z})$. An approximation argument extends this argument to $h \in \mathcal{H}^\infty$, and identifies $h(L)$ as the coanalytic Toeplitz operator with symbol $h(\overline{z})$. $\square$

Corollary 2.4. *The space $\mathcal{A}^\infty$ is invariant for the coanalytic Toeplitz operators.*

Proof. It suffices to note that $f \in \mathcal{A}^\infty$ if and only if $f \in \mathcal{H}_\alpha^2$ for all $\alpha \geq 0$, and apply Proposition 2.3. $\square$

Remark 2.5. Other spaces invariant for the coanalytic Toeplitz operators include the Hölder classes Λ_a^α for $\alpha \in (0,1)$, and the Gevrey classes $\mathcal{G}_\beta$ for $\beta \in (0,1]$ (this claim follows from (2.7) below and Proposition 2.3). Notably, the disk algebra $\mathcal{A}$ fails to be invariant for certain coanalytic Toeplitz operators. If $\mathcal{T}_{\overline{h}} : \mathcal{A} \to \mathcal{A}$ is indeed bounded, then the adjoint operator under the Cauchy duality (see Section 2.4 below) is readily identified with the multiplication operator $g \mapsto hg$ on the dual space $\mathcal{A}^*$, which turns out to be the space $\mathcal{K}$ of Cauchy transforms of Borel measures on $\mathbb{T}$. However, according to [10, Chapter 6], not every bounded analytic function h is a multiplier on $\mathcal{K}$.

2.3. **Integral norms.** At a later stage, it will be convenient to replace the norms on $\mathcal{H}_\alpha^2$ and $\mathcal{GH}_{\beta,c}^2$ in (2.3) and (2.4) by integral expressions.

For $\alpha < 0$, we can identify $\mathcal{H}_\alpha^2$ with the following space $\mathcal{P}^2(dA_\alpha)$ of analytic functions:

$$\mathcal{P}^2(dA_\alpha) := \Big\{ f \in Hol(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^2 dA_\alpha(z) < \infty \Big\}, \tag{2.5}$$

where

$$dA_\alpha(z) := (1-|z|)^{-\alpha-1} dA(z)$$

and dA is the area measure on $\mathbb{D}$. The square root of the integral expression in (2.5) defines a norm which is equivalent to the norm $\|\cdot\|_{\mathcal{H}^2_\alpha}$ in (2.3) (see [19, Chapter 1] for a proof), and the spaces $\mathcal{H}^2_\alpha$ and $\mathcal{P}^2(dA_\alpha)$ coincide as sets. For $\alpha \in [0,2)$ we instead have that

$$\begin{aligned} \|f\|^2_{\mathcal{H}^2_\alpha} &\simeq |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1-|z|)^{1-\alpha} dA(z) \\ &= |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA_{\alpha-2}(z). \end{aligned} \tag{2.6}$$

Similar equivalences exist also for $\alpha \geq 2$ and involve higher derivatives of f, but we shall not need them. Note that by (2.6) the Hölder class Λ^γ_a is contained in $\mathcal{H}^2_\alpha$ whenever $\gamma > \alpha/2$. The containment is readily established by using the Hardy-Littlewood characterization of Λ^γ_a as the space of analytic functions f in $\mathbb{D}$ which satisfy the growth condition $|f'(z)| \leq C_f(1-|z|)^{\gamma-1}$ (see [39, Chapter VII, Section 5]).

The corresponding result for $\mathcal{GH}^2_{\beta,c}$ is only a bit more complicated, and looks as follows. We have the readily verified set equality

$$\mathcal{G}_\beta = \bigcup_{c>0} \mathcal{GH}^2_{\beta,c}. \tag{2.7}$$

If we set $\widetilde{\beta} := \frac{\beta}{1-\beta}$ for $\beta \in (0,1)$, and

$$\mathcal{E}_{\widetilde{\beta},c}(z) := \exp\big(-c(1-|z|)^{-\widetilde{\beta}}\big), \quad z \in \mathbb{D}, \tag{2.8}$$

then we have the set equality

$$\bigcup_{c<0} \mathcal{GH}^2_{\beta,c} = \bigcup_{c>0} \mathcal{P}^2(\mathcal{E}_{\widetilde{\beta},c} dA) \tag{2.9}$$

where

$$\mathcal{P}^2(\mathcal{E}_{\widetilde{\beta},c} dA) := \Big\{ f \in Hol(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^2 \mathcal{E}_{\widetilde{\beta},c}(z) dA(z) < \infty \Big\}. \tag{2.10}$$

The square root of the integral expression on the right-hand side of (2.10) is the norm on $\mathcal{P}^2(\mathcal{E}_{\widetilde{\beta},c}dA)$. The set equality (2.9) follows from a rather messy computation of the moments of the weights $\mathcal{E}_{\widetilde{\beta},c}$, which we prefer to skip. See [32, Lemma 5.7] for details. Moreover, in (2.9), for any fixed index $c < 0$, there exists $c' > 0$ such that $\mathcal{GH}^2_{\beta,c}$ is continuously embedded in $\mathcal{P}^2(\mathcal{E}_{\widetilde{\beta},c'}dA)$, and a converse statement with the roles of the two spaces reversed holds too.

2.4. **Cauchy duality.** This concept is a convenient tool in the study of containment of regular functions in K_θ and $\mathcal{H}(b)$. Let X be a topological space of analytic functions on $\mathbb{D}$. In our presentation, usually X will consist of bounded functions, contain the polynomials, and we shall often be able to reduce to the case of X being a Hilbert space. It is sometimes the case that X admits a dual space X^* which is itself a space of analytic functions on $\mathbb{D}$, with the duality pairing between the spaces realized by

$$\langle f, g \rangle := \lim_{r \to 1} \int_{\mathbb{T}} f(r\zeta)\overline{g(r\zeta)} dm(\zeta), \quad f \in X, g \in X^*. \tag{2.11}$$

Such duality pairings are called *Cauchy pairings*, and X^* is then the *Cauchy dual* to X. In the case that both f and g are members of $\mathcal{H}^2$, the definition (2.11) reduces to

$$\langle f, g \rangle = \int_{\mathbb{T}} f\overline{g}\, dm.$$

This important relation shows that $\langle f, g \rangle = 0$ if and only if the two functions f and g are orthogonal in $\mathcal{H}^2$. It is also the reason for why we use the same notation for the Cauchy pairing and the $L^2(dm)$-inner product.

We shall not dwell on matters such as conditions for the existence of a Cauchy dual. In the particular examples X which will appear in this article, the Cauchy dual X^* will be readily identified as a well-known Banach space of functions. For instance, the disk algebra $X = \mathcal{A}$ has as its Cauchy dual the space $X^* = \mathcal{K}$ of Cauchy transforms of Borel measures on $\mathbb{T}$ (see [10, Chapter 4] for details). If X is a Hilbert space, then so is X^*, and the weak-star topology on X^* coincides with the weak topology on X^*. In particular, the space $\mathcal{H}^2_\alpha$ defined in (2.3) is the Cauchy dual to $\mathcal{H}^2_{-\alpha}$, and the space $\mathcal{GH}^2_{\beta,c}$ defined in (2.4) is the Cauchy dual to $\mathcal{GH}^2_{\beta,-c}$.

3. Regular functions in $\mathcal{H}(b)$

This section constitutes the core of the article. Our developments will lead to a proof of the theorem stated in Section 1.

3.1. **Continuous functions in model spaces.** Before treating the general case, it is necessary to understand the containment problem in the special case that $b = \theta$ is inner, and so when $\mathcal{H}(b)$ reduces to a model space K_θ defined in (1.7).

If we have $\theta(\lambda) = 0$ for some $\lambda \in \mathbb{D}$, then K_θ contains the rational function

$$k_\lambda(z) = \frac{1}{1-\overline{\lambda}z} \in K_\theta. \tag{3.1}$$

Thus K_θ contains a non-zero function of ultimate regularity, since k_λ is holomorphic in a neighbourhood of the closure of $\mathbb{D}$. In the case that θ does not vanish in $\mathbb{D}$, we have to work much harder to exhibit a non-zero function in $K_\theta \cap \mathcal{A}$.

Example 3.1. Consider the singular inner function

$$\theta(z) = \exp\Big(-\frac{1+z}{1-z}\Big), \quad z \in \mathbb{D}. \tag{3.2}$$

It is quite a brain teaser to find a non-zero function in K_θ which is simultaneously a member of the disk algebra $\mathcal{A}$. Here is one example:

$$f(z) := \frac{\theta(z)(1-z) - \theta(0) - \theta'(0)z + \theta(0)z}{z^2}, \quad z \in \mathbb{D}. \tag{3.3}$$

To see that $f \in \mathcal{A}$, note that θ is holomorphic in $\mathbb{C} \setminus \{1\}$, and so f extends to a continuous function on $\mathbb{T} \setminus \{1\}$. Moreover, since θ is bounded in $\mathbb{D} \cup \mathbb{T}$, it is clear from the formula (3.3) that

$$\lim_{z \to 1} f(z) = f(1) := -\theta'(0).$$

The singularity at $z = 0$ in the expression for f is easily seen to be removable. Thus $f \in \mathcal{A}$. To see that $f \in K_\theta$, it suffices to verify that $f \perp \theta\mathcal{H}^2$. Recall that $L = \mathcal{T}_{\overline{z}}$ denotes the backward shift operator in (2.1). One may perform a simple computation to see that

$$f = (L^2 - L)\theta.$$

If $h \in \mathcal{H}^2$, we have

$$\big\langle f, \theta h \big\rangle = \big\langle \theta, (z^2 - z)\theta h \big\rangle = \big\langle 1, (z^2 - z)h \big\rangle = 0,$$

where we used that multiplication by θ is an isometry in $\mathcal{H}^2$. One may also see that $f \in K_\theta$ by noting that $f = \mathcal{T}_{\overline{z}-1}L\theta$ and recalling the well-known facts that $L\theta \in K_\theta$ and that K_θ is invariant for coanalytic Toeplitz operators.

Although it might not be immediately clear, setting instead $f := (L-1)^n L\theta$ for some integer $n > 1$ will lead to f having a higher degree of regularity on $\mathbb{T}$. This observation will eventually lead us to a theorem of Dyakonov-Khavinson from [12] on characterization of the non-triviality of the intersection $K_\theta \cap \mathcal{A}^\infty$.

In general, we have the following result from [1], which we will refer to as *Aleksandrov's theorem.*

Theorem 3.2. *For every inner function θ, the intersection $K_\theta \cap \mathcal{A}$ is norm-dense in K_θ. In particular, $K_\theta \cap \mathcal{A}$ always contains a non-zero function.*

The proofs of this theorem which are known to the author all go by duality. One proof, given in the book [10, Chapter 8.5] by Cima, Matheson and Ross, is based on Aleksandrov's follow-up work [2], and establishes slightly more (namely weak-star density of $K_\theta \cap \mathcal{A}$ in $K_\theta \cap \mathcal{H}^\infty$). The following proof has been shown to the author by Aleman.

Proof of Theorem 3.2. Assume that a function $g \in K_\theta$ is orthogonal to all functions in $K_\theta \cap \mathcal{A}$. We must show that $g \equiv 0$. The trick is to interpret g as an element of the Cauchy dual $\mathcal{K} = \mathcal{A}^*$ which is the space of Cauchy transforms of finite Borel measures on $\mathbb{T}$ (see [10] for details, the exact structure of the space will not be important in the proof). Then the assumed orthogonality implies that g annihilates the linear manifold $K_\theta \cap \mathcal{A}$. By the Hahn-Banach separation theorem, we have that g lies in the weak-star closure of the convex subset $\{\theta h\}_{h\in\mathcal{H}^2}$ in $\mathcal{K}$. Indeed, in the contrary case the separation theorem gives us an $f \in \mathcal{A}$ which satisfies $\langle f, g\rangle \neq 0$ and $\langle f, \theta h\rangle = 0$ for all $h \in \mathcal{H}^2$. The second expression is equivalent to $f \in K_\theta \cap \mathcal{A}$, and then by the first expression we reach a contradiction to our orthogonality assumption on g. Thus g indeed lies in the weak-star closure of $\{\theta h\}_{h\in\mathcal{H}^2}$, which equals the intersection of all weak-star closed subsets of $\mathcal{K}$ containing $\{\theta h\}_{h\in\mathcal{H}^2}$. One such set is

$$\theta(\mathcal{K}) := \{f \in \mathcal{K} : f/\theta \in N^+\},$$

where N^+ denotes the usual Smirnov class of quotients of bounded analytic functions on $\mathbb{D}$ with outer denominator. See [10, Theorem 8.5.4] for a proof of this claim. In particular, $g \in \theta(\mathcal{K})$, so in fact $g = \theta r$ for some $r \in \mathcal{H}^2$. But then $g \in K_\theta \cap \theta\mathcal{H}^2 = \{0\}$, and the proof is complete. □

As for sharpness, it was observed by the author and Limani in [28] that Aleksandrov's theorem holds with $\mathcal{A}$ replaced by the space $\mathcal{U}$ of *uniformly* convergent Taylor series in $\mathbb{D}$. The proof is as above, but with the critical part (namely, the application of [10, Theorem 8.5.4]) replaced by a corresponding property of the Cauchy dual of $\mathcal{U}$. Conversely, it is known that $\mathcal{A}$ in Theorem 3.2 cannot be replaced by any class of functions defined by their *modulus of continuity* (see [30]).

3.2. **Smooth functions in model spaces.** Constructive approaches to Aleksandrov's Theorem 3.2 are lacking, but replacing $\mathcal{A}$ by the smaller class of $\mathcal{A}^\infty$ leads to a setting in which constructions are available. Their essence is contained in Example 3.1. There, we had

$$f = (L^2 - L)\theta = P_+(\overline{z}^2 - \overline{z})\theta.$$

We computed explicitly that $f \in K_\theta \cap \mathcal{A}$. Without carrying out the computation, one may expect that f satisfies some additional regularity by noticing that multiplication by $\overline{z}^2 - \overline{z}$ removed the discontinuity of θ at $z = 1$. By exchanging the symbol $\overline{z}^2 - \overline{z}$ by $(\overline{z} - 1)^n\overline{z}$, which has a zero at $z = 1$ of higher order, we may in fact ensure that $(\overline{z} - 1)^n\overline{z}\theta$ is continuously differentiable any finite number of times on $\mathbb{T}$. Then $P_+(\overline{z}^{n+1} - \overline{z}^n)\theta$ will be in $\mathcal{A}^n$ for some $n \geq 1$.

Based on this idea, one is lead to wonder what closed sets $E \subset \mathbb{T}$ are zero sets of analytic functions m with a high degree of smoothness. The question has been answered by Carleson in [9], who showed that $E \subset \mathbb{T}$ is a zero set of a function $m \in \mathcal{A}^n$, $n \geq 1$, if and only if E is a closed set of zero Lebesgue measure and satisfies

$$\sum_\ell |\ell| \log(1/|\ell|) < \infty, \tag{3.4}$$

where $\{\ell\}$ is the set of maximal open arcs complementary to E in $\mathbb{T}$. Sets satisfying (3.4) are nowadays known as *Carleson sets*, and in our presentation they will appear both as sets of zero and positive Lebesgue measure. Taylor and Williams in [36] later extended Carleson's argument to $m \in \mathcal{A}^\infty$. Another construction is given in the book [19, Lemma 7.11] by Hedenmalm, Korenblum and Zhu. More precisely, given a Carleson set E, there exists an outer function $m \in \mathcal{A}^\infty$ which satisfies for any $A > 0$ the estimate

$$|m(z)| = O\big((\mathrm{dist}(z, E))^A\big), \quad z \in \mathbb{D} \cup \mathbb{T} \tag{3.5}$$

and a similar estimate holds with m replaced by any derivative $m^{(n)}$. From this, we derive the sufficient condition of the following theorem due to Dyakonov and Khavinson from [12].

Theorem 3.3. *Let $\theta = BS_\nu$ be an inner function, where B is a Blaschke product and S_ν is singular inner. The following two statements are equivalent.*

(i) The intersection $\mathcal{A}^\infty \cap K_\theta$ contains a non-zero function.
(ii) Either $\theta(\lambda) = 0$ for some $\lambda \in \mathbb{D}$, or $\nu(E) > 0$ for some Carleson set E of Lebesgue measure zero.

Proof of the sufficiency part $(ii) \Rightarrow (i)$ *of Theorem* 3.3. If $\theta(\lambda) = 0$, then K_θ contains the smooth function k_λ in (3.1). If $\nu(E) > 0$ for a Carleson set of zero Lebesgue measure, then we may suppose that $\theta = S_{\nu|E}$, where $\nu|E$ is the restriction of ν to E. This follows since $K_{S_{\nu|E}} \subset K_\theta$. Under this assumption, from (1.4) we deduce that

$$|\theta^{(n)}(z)| = O\big(\mathrm{dist}(z, E)^{-2n}\big), \quad z \in (\mathbb{D} \cup \mathbb{T}) \setminus E.$$

If m satisfies (3.5), then it readily follows that $\overline{mz}\theta \in C^\infty(\mathbb{T})$. By Toeplitz invariance of K_θ and smoothness preservation of P_+ (recall the first paragraph of Section 2.1) we obtain

$$P_+\overline{mz}\theta = \mathcal{T}_{\overline{m}} L\theta \in K_\theta \cap \mathcal{A}^\infty.$$

The fact that the above smooth function is non-zero is a consequence of m being outer, which ensures injectivity of $\mathcal{T}_{\overline{m}}$. □

The implication $(i) \Rightarrow (ii)$ is much more difficult to prove. Dyakonov and Khavinson prove the implication by using the Korenblum-Roberts theorem on cyclicity of singular inner functions in Bergman-type spaces. We will describe this result in a short while, and also the related notion of *splitting sequences* which are necessary to prove a generalization of the Dyakonov-Khavinson result for the general class of $\mathcal{H}(b)$-spaces.

Remark 3.4. Carleson sets, and their generalizations, appear in numerous important recent works on the subject of model spaces and singular inner functions. They appear in connection to the question of existence of univalent maps in model spaces in the article [6] by Baranov and Fedorovskiy, and in a follow-up article [5] by Baranov, Belov, Borichev and Fedorovskiy which fully answers the question. They apper also in Ivrii's description of the structure of the critical sets of inner functions with derivatives in the Nevanlinna class. His deep results are found in [20]. Nicolau and Ivrii describe further interesting applications in [21].

3.3. **Khrushchev's theorems and regular functions in $\mathcal{H}(b)$.** Sufficient conditions for $\mathcal{H}(b)$ to contain a regular function follow from fundamental results of Khrushchev on the projection operator $P_+ : L^2(dm) \to \mathcal{H}^2$. In [22], he proved the following two striking theorems.

Theorem 3.5. *Let E be any subset of $\mathbb{T}$ of positive Lebesgue measure. There exists a measurable function $k \in L^2(\mathbb{T})$ which lives only on E for which we have*

$$P_+k \in \mathcal{A} \setminus \{0\}.$$

Theorem 3.6. *Let E be a subset of $\mathbb{T}$ of positive Lebesgue measure. There exists a measurable function $k \in L^2(\mathbb{T})$ which lives only on E for which we have*

$$P_+k \in \mathcal{A}^\infty \setminus \{0\}$$

if and only if E contains a subset of positive Lebesgue measure satisfying the Carleson condition (3.4).

We refer the reader to [22], or alternatively to [18], for proofs of the above theorems. The original proofs are non-constructive, but a constructive proof of Theorem 3.6 has been given in [29], using a technique not much different than the one appearing in the proof of the implication $(ii) \Rightarrow (i)$ of the Dyakonov-Khavinson Theorem 3.3 above. The author knows of no constructive proof of the first of the above theorems.

Corollary 3.7. *The intersection $\mathcal{H}(b) \cap \mathcal{A}$ always contains a non-zero function.*

Proof. If the inner factor θ_b is non-trivial, then the claim follows from Aleksandrov's Theorem 3.2, since $K_{\theta_b} \subset \mathcal{H}(b)$. In the other case $\Delta_b = \sqrt{1 - |b|^2}$ is not the zero function on $\mathbb{T}$. Choose $c > 0$ so that

$$E := \{\zeta \in \mathbb{T} : \Delta_b(\zeta) > c\}$$

has positive Lebesgue measure. By Theorem 3.5, there exists a function $k \in L^2(dm)$ living only on E for which $P_+k \in \mathcal{A} \setminus \{0\}$. If we set

$$k_0 = \begin{cases} k(\zeta)/\Delta_b(\zeta) & \zeta \in E \\ 0 & \zeta \notin E \end{cases}$$

then since $\Delta_b > c$ on E, we have $k_0 \in L^2(dm)$ and

$$k_0\Delta_b = k.$$

By Proposition 2.2, we have $P_+k = P_+k_0\Delta_b \in \mathcal{H}(b) \cap \mathcal{A}$, and the proof is complete. □

The same proof as the one given Corollary 3.7 does not work in the smooth case, since the superlevel set E appearing the proof might not have the necessary structure (namely, contain a Carleson set of positive measure) to apply Khrushchev's Theorem 3.6. Instead, a characterization of the non-triviality of the intersection $\mathcal{H}(b) \cap \mathcal{A}^\infty$ looks as follows. It is the main new result of this article, and constitutes a generalization of the Dyakonov-Khavinson Theorem 3.3 to the context of de Branges-Rovnyak spaces.

Theorem 3.8. *Let $b = BS_\nu b_0$ be the Nevanlinna factorization of b. The following two statements are equivalent.*

(i) The intersection $\mathcal{H}(b) \cap \mathcal{A}^\infty$ contains a non-zero function.

(ii) Either $b(\lambda) = 0$ for some $\lambda \in \mathbb{D}$, or $\nu(E) > 0$ for some Carleson set E of zero Lebesgue measure, or $\int_E \log \Delta_b \, dm > -\infty$ for some Carleson set E of positive Lebesgue measure.

As in the case of Theorem 3.3, we first prove the easier sufficiency part, leaving the more difficult necessity part to be proved below.

Proof of the sufficiency part $(ii) \Rightarrow (i)$ of Theorem 3.8. Since $K_{\theta_b} \subset \mathcal{H}(b)$, if either $b(\lambda) = 0$ or $\nu(E) > 0$ for some Carleson set of Lebesgue measure zero, then the result follows from the already proved part $(ii) \Rightarrow (i)$ of Theorem 3.3. It remains to show how the condition $\int_E \log \Delta_b \, dm > -\infty$ for some Carleson set E of positive Lebesgue measure implies the existence of a non-zero function in $\mathcal{H}(b) \cap \mathcal{A}^\infty$. Elementary Hardy space theory (see, for instance, [17, Chapter II]) ensures that there exists a bounded outer function h which has boundary values satisfying

$$|h(\zeta)| = \Delta_b(\zeta)$$

for almost every $\zeta \in E$. Furthermore, by Khrushchev's Theorem 3.6 there exists a function $k \in L^2(dm)$ living only on the Carleson set E for which we have $P_+ k \in \mathcal{A}^\infty \setminus \{0\}$. Note that

$$g := \mathcal{T}_{\overline{h}} P_+ k = P_+ \overline{h} k = P_+ k u \Delta_b,$$

where u is a function which is unimodular on E, satisfies $u = \overline{h}/\Delta_b$ on that set, and vanishes elsewhere on $\mathbb{T}$. Hence $ku \in L^2(dm)$. By Proposition 2.2, we have $g \in \mathcal{H}(b)$. Moreover, g is non-zero, since $P_+ k$ is non-zero, and h is outer, so that $\mathcal{T}_{\overline{h}}$ is injective. It remains to note that $g \in \mathcal{A}^\infty$, which is consequence of $P_+ k \in \mathcal{A}^\infty$ and Corollary 2.4. □

3.4. **Applications of cyclicity of singular inner functions.** Having proved the implications $(ii) \Rightarrow (i)$ in Theorem 3.3 and Theorem 3.8, we turn to the reverse implications. In the first theorem, Dyakonov and Khavinson proved the implication by exploiting cyclicity of certain singular inner functions. We will discuss their proof in detail in order to emphasize its similiarity to the way in which we will prove the necessity part $(i) \Rightarrow (ii)$ in Theorem 3.8.

Let Y be a topological space of analytic functions on $\mathbb{D}$. We shall assume, for convenience, that Y contains all bounded analytic functions. We will say that a function $g \in Y$ is *cyclic* if there exists a sequence of polynomials $\{p_n\}_n$ such that

$$gp_n \to 1$$

in the topology of Y. If the multiplication operator $M_z = \mathcal{T}_z : f(z) \mapsto zf(z)$ acts on Y, and the polynomials are dense in Y, then our notion of cyclicity coincides with the usual notion of g being a cyclic vector for M_z.

Here is the way in which cyclicity of singular inner functions is relevant. The proof is inspired by the argument of Dyakonov and Khavinson from [12].

Lemma 3.9. *Assume that X is a Banach space of analytic functions on $\mathbb{D}$ which is invariant for the backward shift operator L, that it admits a Cauchy dual X^*, and that the singular inner function θ is cyclic in the weak-star topology on X^*. Then $K_\theta \cap X = \{0\}$.*

Conversely, if the linear manifold $\{\theta h\}_{h \in \mathcal{H}^2}$ is contained but not weak-star dense in X^*, and $X \subset \mathcal{H}^2$, then $K_\theta \cap X$ contains a non-zero function.

Proof. Suppose that the singular inner function θ is cyclic in the weak-star topology of X^*, and let $g \in K_\theta \cap X$. We must show that $g \equiv 0$. Let $\{p_n\}_n$ be a sequence of polynomials such that $\theta p_n \to 1$ in the weak-star topology on X^*. Then, we have

$$g(0) = \langle g, 1 \rangle = \lim_n \langle g, \theta p_n \rangle = 0$$

where the last equality follows since $g \in K_\theta$ and the duality pairing coincides here with the inner product on $\mathcal{H}^2$. For any positive integer k, we have that $L^k g \in K_\theta \cap X$, by the L-invariance of X and K_θ. Thus the same argument shows that $L^k g(0) = \widehat{g}(k) = 0$, where $\widehat{g}(k)$ is the k:th Fourier coefficient of g. Thus $g \equiv 0$.

We prove the converse statement. Recall that the dual space of X^* equipped with the weak-star topology is X itself. It remains to note that if $\{\theta h\}_{h \in \mathcal{H}^2}$ is not weak-star dense in X^*, then by the Hahn-Banach separation theorem, a non-zero $g \in X$ exists which satisfies $\langle g, \theta h \rangle = 0$ for all $h \in \mathcal{H}^2$. By definition of K_θ, this means that $g \in K_\theta \cap X$. □

Remark 3.10. We shall not use the second statement in Lemma 3.9 in our proofs. However, note that if X is a Hilbert space, then the two statements combine to show that $\mathcal{K}_\theta \cap X = \{0\}$ if and only θ is cyclic in the norm topology on X^*. We will come back to this observation in Section 4.

The implication $(i) \Rightarrow (ii)$ in Theorem 3.3 now follows from the deep works of Korenblum in [23] and [24], or Roberts in [33]. Recall the definition of the space $\mathcal{H}^2_\alpha$ in (2.3), $\mathcal{P}^2(dA_\alpha)$ in (2.5) and that these spaces in fact coincide as sets for $\alpha < 0$, with the corresponding norms being equivalent.

The following is the Korenblum-Roberts cyclicity theorem.

Theorem 3.11. *If $\alpha < 0$, then the singular inner function S_ν is cyclic in $\mathcal{P}^2(dA_\alpha)$ if and only if*

$$\nu(E) = 0$$

for all Carleson sets E of zero Lebesgue measure.

Korenblum derived the theorem in [25] as a consequence of his more general results in [23] and [24]. Roberts gave an independent proof in [33].

Proof of the necessity part $(i) \Rightarrow (ii)$ of Theorem 3.3. Assume that (ii) does not hold, so that $\theta = S_\nu$ for ν which satisfies $\nu(E) = 0$ for all Carleson sets of zero Lebesgue measure. By Theorem 3.11, S_ν is cyclic in $X^* = \mathcal{P}^2(dA_\alpha) = \mathcal{H}^2_\alpha$ for every $\alpha < 0$, and so an application of Lemma 3.9 to $X = \mathcal{H}^2_{-\alpha}$ shows that $K_\theta \cap \mathcal{H}^2_{-\alpha} = \{0\}$ for all $\alpha < 0$. In particular, $K_\theta \cap \mathcal{A}^\infty = \{0\}$. □

Remark 3.12. The above proof shows that the condition on ν implies the stronger statement $K_\theta \cap \mathcal{H}^2_{-\alpha} = \{0\}$ for any $\alpha < 0$. By the observation made in Section 2.3 regarding containment of Hölder class Λ^γ_a in $\mathcal{H}^2_\alpha$ whenever $\gamma > \alpha/2$, we deduce that the condition on ν ensures also $K_\theta \cap \Lambda^\gamma_a = \{0\}$ for every $\gamma \in (0, 1]$.

Remark 3.13. In the same article [12], Dyakonov and Khavinson mention also an application of the existence of cyclic singular inner functions in the Bloch space $\mathcal{B}$ consisting of functions satisfying $\sup_{z \in \mathbb{D}}(1 - |z|)|f'(z)| < \infty$. The Bloch space is the dual of the Sobolev-type space $W^{1,1}_a$ consisting of all analytic functions in $\mathbb{D}$ satisfying $\int_{\mathbb{D}} |f'| dA < \infty$. By results of, for instance, [4], there exists a singular inner function S_ν which is weak-star cyclic in $\mathcal{B}$. Then $K_{S_\nu} \cap W^{1,1}_a = \{0\}$, by Lemma 3.9. The triangle inequality shows that the Wiener algebra $\mathcal{W}$ of Taylor series satisfying

$$\|f\|_{\mathcal{W}} := \sum_{k \geq 0} = |f_k| < \infty$$

is contained in $W^{1,1}_a$. Thus, in particular, there exist θ for which $K_\theta \cap \mathcal{W} = \{0\}$. This observation has been shown to the author by Limani.

3.5. **Splitting sequences and their applications.** In the case that b is an outer function, singular inner functions in the previous proofs are replaced by *splitting sequences.*

Definition 3.14. Let $Y_{\mathbb{D}}$ be a topological space of functions on $\mathbb{D}$, and $Y_{\mathbb{T}}$ be a topological space of functions on $\mathbb{T}$, both of which contain the analytic polynomials. A sequence $\{p_n\}_n$ of analytic polynomials is $(Y_{\mathbb{D}}, Y_{\mathbb{T}})$-*splitting* if we have that

$$p_n \to 1 \text{ in } Y_{\mathbb{D}}$$

and

$$p_n \to 0 \text{ in } Y_{\mathbb{T}},$$

with convergence in the sense of the corresponding topologies.

Note that the existence of a $(Y_{\mathbb{D}}, Y_{\mathbb{T}})$-splitting sequence is equivalent to the vector $(1,1) \in Y_{\mathbb{D}} \oplus Y_{\mathbb{T}}$ being cyclic for the operator $M_z \oplus M_z$, where $M_z f(z) = zf(z)$ (assuming that $Y_{\mathbb{D}}$ and $Y_{\mathbb{T}}$ are invariant for M_z, and that the polynomials are dense in both spaces).

Example 3.15. Khrushchev in [22] proved that if $dm|E$ is the restriction of the Lebesgue measure on $\mathbb{T}$ to a measurable subset $E \subset \mathbb{T}$ of positive Lebesgue measure, and $\alpha < 0$, then a $\big(\mathcal{P}^2(dA_\alpha), L^2(dm|E)\big)$-splitting sequence exists if and only if E contains no Carleson sets of positive Lebesgue measure (recall the definition in (3.4)).

Example 3.16. Let $\mathcal{P}^2(\mu)$ be the closure of analytic polynomials in the Lebesgue space $L^2(\mu)$, where μ is a non-negative finite Borel measure compactly supported in the plane $\mathbb{C}$. In the case that μ has the structure

$$\begin{aligned} d\mu &= d\mu|\mathbb{D} + d\mu|\mathbb{T} \\ &= G(1-|z|)dA(z) + w(z)dm(z) \end{aligned} \tag{3.6}$$

where G is integrable on $[0,1)$ and w integrable on $\mathbb{T}$, the existence of a $\big(\mathcal{P}^2(\mu|\mathbb{D}), L^2(\mu|\mathbb{T})\big)$-splitting sequence is equivalent to the existence of a decomposition

$$\mathcal{P}^2(\mu) = \mathcal{P}^2(\mu|\mathbb{D}) \oplus L^2(\mu|\mathbb{T}), \tag{3.7}$$

where $\mathcal{P}^2(\mu)$ and $\mathcal{P}^2(\mu|\mathbb{D})$ are the closures of polynomials in the corresponding Lebesgue spaces. The *splitting problem* asks to characterize the pairs (G, w) in (3.6) for which decomposition (3.7) holds. Important works on the splitting problem from the 1980s and 1990s include the theorem of Volberg from [37] (with a different exposition by Volberg and Jöricke available in [38]) which deals with very rapidly decreasing G, and the article by Kriete and MacCluer [26] which studies a more general case.

We shall soon see more examples of splitting sequences. First, we explain how they fit into our study. In a typical application, $Y_{\mathbb{T}}$ is a weighted Lebesgue space $\mathcal{L}^2(\Delta_b\, dm)$, where Δ_b is as in (1.9), and $Y_{\mathbb{D}} = X^*$ is the Cauchy dual of a space X of analytic functions on $\mathbb{D}$. The following result is a counterpart to Lemma 3.9.

Lemma 3.17. *Assume that b is outer, and X is a space of analytic functions on $\mathbb{D}$ which is invariant for the coanalytic Toeplitz operators. If there exists a $(X^*, L^2(\Delta_b dm))$-splitting sequence, then*

$$\mathcal{H}(b) \cap X = \{0\}.$$

Proof. Similarly to before, we let $g_0 \in \mathcal{H}(b) \cap X$ and show that $g_0 \equiv 0$. By the assumed invariance of the space X and by Proposition 2.1, we have that

$$g := \mathcal{T}_{\overline{b}} g_0 \in \mathcal{H}(b) \cap X.$$

Note that $g \equiv 0$ if and only if $g_0 \equiv 0$, since b is outer and so $\mathcal{T}_{\overline{b}}$ is injective. Thus it suffices to show that $g \equiv 0$. Let $\{p_n\}_n$ be a $(X^*, L^2(\Delta_b dm))$-splitting sequence. Since $p_n \to 1$ in X^*, we have

$$g(0) = \langle g, 1 \rangle = \lim_n \langle g, p_n \rangle.$$

According to Proposition 2.1, we have

$$g = P_+ \Delta_b k$$

for some $k \in L^2(dm)$. Since the polynomials p_n are analytic, we have

$$\lim_n \langle g, p_n \rangle = \lim_n \langle \Delta_b k, p_n \rangle = \lim_n \int_{\mathbb{T}} k \overline{p_n} \Delta_b \, dm = 0$$

where the last equality holds since $p_n \to 0$ in $L^2(\Delta_b dm)$ and $k \in L^2(dm) \subset L^2(\Delta_b dm)$. Thus $g(0) = 0$. Since for any $k \geq 1$ we have that $L^k g$ satisfies the same assumptions as g, we conclude that the Fourier series of g vanishes, and consequently $g \equiv 0$. □

The equivalent of the Korenblum-Roberts Theorem 3.11 for splitting sequences is the following result of the author and Bergqvist from [7] generalizing Khrushchev's result discussed in Example 3.15.

Proposition 3.18. *Let w be a non-negative integrable weight on $\mathbb{T}$, and let $\alpha < 0$. There exists a $\big(\mathcal{P}^2(dA_\alpha), L^2(w\,dm)\big)$-splitting sequence if and only if*

$$\int_E \log w\,dm = -\infty$$

for every Carleson set E of positive Lebesgue measure.

The above result was conjectured by Kriete and MacCluer in [26].

We are now ready to prove the remaining implication $(i) \Rightarrow (ii)$ in Theorem 3.8. This will complete the proof of the main new result of this article.

Proof of the necessity part $(i) \Rightarrow (ii)$ *of Theorem* 3.8. Assume that (ii) does not hold. We will show the stronger statement that if $\alpha < 0$ and $g \in \mathcal{H}(b) \cap \mathcal{H}^2_{-\alpha}$, then $g \equiv 0$. Note that the inner factor $\theta_b = BS_\nu$ of b fails to satisfy condition (ii) in the Dyakonov-Khavinson Theorem 3.3. If $g \in K_{\theta_b}$, then by Remark 3.12 we have $g \equiv 0$, and the proof is complete. We may thus assume that $g \notin K_{\theta_b}$, so that for $g_0 := \mathcal{T}_{\overline{b}} g$ we have $g_0 \equiv 0$ if and only if $g \equiv 0$, since the kernel of $\mathcal{T}_{\overline{b}}$ equals K_{θ_b}. By Proposition 2.1 and the Toeplitz invariance in Proposition 2.3, we have $g_0 \in \mathcal{H}(b_0) \cap \mathcal{H}^2_{-\alpha}$. Here b_0 is the outer part of b. By our assumption, Proposition 3.18 applies to the weight

$$w = \Delta_b = \sqrt{1 - |b|^2} = \sqrt{1 - |b_0|^2}$$

and provides us with a $(\mathcal{P}^2(dA_\alpha), L^2(\Delta_b\,dm))$-splitting sequence. Since b_0 is outer, by Lemma 3.17 we obtain that $g_0 \in \mathcal{H}(b_0) \cap \mathcal{H}^2_{-\alpha} = \{0\}$. □

Remark 3.19. As in Remark 3.12, the proof actually shows that if the three conditions in (ii) of Theorem 3.8 are all not satisfied, then $\mathcal{H}(b) \cap \Lambda^\gamma_a = \{0\}$ for every $\gamma \in (0, 1]$.

3.6. **The curious case of the Gevrey classes.** Recall the definition of the Gevrey classes $\mathcal{G}_\beta$ in Section 1. There occurs an interesting change in behaviour at $\beta = 1/2$. We will see that the model space K_θ contains a non-zero function in $\mathcal{G}_{1/2}$ only in the trivial case that θ vanishes at some point in $\mathbb{D}$. On the other hand, $\mathcal{H}(b)$ may contain $\mathcal{G}_\beta$ for any $\beta \in (0, 1)$ even though its symbol b is extreme and does not vanish in $\mathbb{D}$.

The following cyclicity theorem can be established by elementary means. We give only a brief sketch of the proof. For a complete argument, see [32]. The article [13] by El-Fallah, Kellay and Seip proves a much stronger result from which the below proposition also follows.

Proposition 3.20. *Let $c > 0$ and $\mathcal{P}^2(\mathcal{E}_{1,c} dA)$ be the space in* (2.10). *Then every singular inner function θ is cyclic in $\mathcal{P}^2(\mathcal{E}_{1,c} dA)$.*

Sketch of proof. Note that $\mathcal{E}_{1,c}(z) = \exp\big(-c(1 - |z|)^{-1}\big)$ decays exponentially as $|z| \to 1$. From (1.4) we deduce that $\theta^{-\gamma} \in \mathcal{P}^2(\mathcal{E}_{1,c} dA)$ if $\gamma > 0$ is sufficiently small. If $\{p_n\}_n$ is a sequence of polynomials for which $p_n \to \theta^{-\gamma}$ in the norm of $\mathcal{P}^2(\mathcal{E}_{1,c} dA)$, then $\theta^\gamma p_n \to 1$ in $\mathcal{P}^2(\mathcal{E}_{1,c} dA)$, and so θ^γ is cyclic. It remains to note that product of finitely many bounded cyclic functions is cyclic (see, for instance, [32, Lemma 4.2]), and that we may choose $1/\gamma$ to be an integer. □

Corollary 3.21. *If θ is a singular inner function, then $K_\theta \cap \mathcal{G}_{1/2} = \{0\}$.*

Proof. Apply Lemma 3.9, Proposition 3.20 and the continuous containment remark following (2.9). The result is that $K_\theta \cap \mathcal{G}\mathcal{H}^2_{1/2,c} = \{0\}$ for every $c > 0$. The corollary now follows from (2.7). □

We will say a word about non-triviality of the intersection $K_\theta \cap \mathcal{G}_\beta$ for $\beta \in (0, 1/2)$ in Section 4.

In the general case, we have the following result for $\mathcal{H}(b)$-spaces.

Theorem 3.22. *Let $\beta \in [1/2, 1)$. The following two statements are equivalent.*

(i) The intersection $\mathcal{H}(b) \cap \mathcal{G}_\beta$ contains a non-zero function.

(ii) Either $b(\lambda) = 0$ for some $\lambda \in \mathbb{D}$, or there exists an arc I of $\mathbb{T}$ for which $\int_I \log \Delta_b\,dm > -\infty$.

We will prove the implication $(i) \Rightarrow (ii)$ by the use of an appropriate splitting sequence, similarly to the proof of the corresponding implication in Theorem 3.8. On the other hand, the implication $(ii) \Rightarrow (i)$ can be established as a consequence of the Beurling-Malliavin theory.

In [8], Beurling and Malliavin gave a sufficient condition for an oscillating non-negative weight W defined on $\mathbb{R}$ to admit a compactly supported function f with a Fourier transform

$$\widehat{f}(\zeta) := \int_{\mathbb{R}} f(x)e^{-i\zeta x}dx, \quad \zeta \in \mathbb{R}$$

satisfying the pointwise bound

$$|\widehat{f}(\zeta)| \leq W(\zeta), \quad \zeta \in \mathbb{R}.$$

See [8] for the precise formulation of the famous Beurling-Malliavin theorem. The theorem can be proved by elementary means in the case that W is even and decreasing for $\zeta > 0$ (see, for instance, [18, p. 276]).

Proposition 3.23. *Assume that W is an even non-negative function on $\mathbb{R}$, decreasing for $\zeta > 0$. If*

$$\int_{\mathbb{R}} \frac{\log W(\zeta)}{1+\zeta^2} d\zeta > -\infty,$$

then for any interval $I \subset \mathbb{R}$, there exists a non-zero function h supported on I satisfying

$$|\widehat{h}(\zeta)| \leq W(\zeta).$$

In order to prove Theorem 3.22 we will need also a new splitting sequence. Recall the definition of the space $\mathcal{P}^2(\mathcal{E}_{1,c}dA)$ from Section 2.3. The following proposition has been established in [31].

Proposition 3.24. *Let w be a non-negative integrable weight on $\mathbb{T}$, and let $c > 0$. There exists a $\big(\mathcal{P}^2(\mathcal{E}_{1,c}dA), L^2(w\,dm)\big)$-splitting sequence if and only if*

$$\int_I \log w\, dm = -\infty$$

for every arc I of $\mathbb{T}$.

Proof of Theorem 3.22. To prove the implication $(ii) \Rightarrow (i)$, we will use an argument due to Kriete and MacCluer from [26]. In the case that $b(\lambda) = 0$, the rational function in (3.1) is contained in $\mathcal{H}(b)$, so (i) holds. If b does not vanish in $\mathbb{D}$, then it follows from (ii) that an arc exists I on which $\log \Delta_b$ is integrable. We may assume that $I = \{e^{it} : t \in [0, c]\}$ for some small $c > 0$. Let r be a non-zero function supported inside the interval $(0, c)$ which satisfies the spectral decay $|\widehat{r}(\zeta)| \leq \exp(-|\zeta|^\beta)$ for some fixed $\beta \in [1/2, 1)$. Such a function exists by Proposition 3.23. If we define the function $r_{\mathbb{T}}$ supported on I by the equality

$$r_{\mathbb{T}}(e^{it}) := r(t), \quad t \in [0, 2\pi),$$

then its n:th Fourier coefficient $\widehat{r_{\mathbb{T}}}(n)$ satisfies

$$\begin{aligned} |\widehat{r_{\mathbb{T}}}(n)| &= \left| \int_0^{2\pi} r_{\mathbb{T}}(e^{it})e^{-int}dm(e^{it}) \right| \\ &= \left| \int_{\mathbb{R}} r(t)e^{-int}\frac{dt}{2\pi} \right| \\ &= \frac{|\widehat{r}(n)|}{2\pi} \\ &\leq \exp(-|n|^\beta). \end{aligned}$$

Thus $g := P_+ r_{\mathbb{T}} \in \mathcal{G}_\beta \setminus \{0\}$. We may now proceed as in the corresponding part of the proof of Theorem 3.8. Take a bounded outer function h satisfying $|h| = \Delta_b$ on I, and recall from Remark 2.5 that $\mathcal{G}_\beta$ is invariant for coanalytic Toeplitz operators. Then, as before, for some u unimodular on I, we have by Proposition 2.2 that

$$\mathcal{T}_{\overline{h}} g = P_+ \Delta_b u r_{\mathbb{T}} \in (\mathcal{H}(b) \cap \mathcal{G}_\beta) \setminus \{0\}.$$

The converse implication $(i) \Rightarrow (ii)$ is proved similarly to the corresponding implication in Theorem 3.8. Note that it suffices for us to show that $\mathcal{H}(b) \cap \mathcal{G}_{1/2} = \{0\}$, since the spaces $\mathcal{G}_\beta$ decrease when β increases.

If (ii) is not satisfied, then Proposition 3.24 gives us a $\big(\mathcal{P}^2(\mathcal{E}_{1,c}dA), L^2(\Delta_b\, dm)\big)$-splitting sequence, and by Lemma 3.17 and an argument analogous to the one given in the proof of Corollary 3.21, we deduce that $\mathcal{H}(b_0) \cap \mathcal{G}_{1/2} = \{0\}$, where b_0 is the outer factor of b. If $g \in \mathcal{H}(b) \cap \mathcal{G}_{1/2}$, then $g_0 = \mathcal{T}_{\overline{b}} g \in \mathcal{H}(b_0) \cap \mathcal{G}_{1/2}$ by Proposition 2.1 and by coanalytic Toeplitz invariance of the Gevrey classes mentioned in Remark 2.5. So $g_0 \equiv 0$, and hence $g \in \ker \mathcal{T}_{\overline{b}}$. By Corollary 3.21 and the assumption that b does not vanish in $\mathbb{D}$, we conclude that no non-zero function in the class $\mathcal{G}_{1/2}$ lies in $K_{\theta_b} = \ker \mathcal{T}_{\overline{b}}$, θ_b being the (singular) inner factor of b. Thus $g \equiv 0$, and the proof is complete. □

4. Comments and open problems

We end the article by explicitly stating a few open problems, solutions of which would further improve our understanding of the containment problem.

4.1. **Construction of a function in $\mathcal{K}_\theta \cap \mathcal{A}$ for general θ.** In Example 3.1 we gave an explicit example of a non-zero function in $K_\theta \cap \mathcal{A}$ for the singular inner function θ corresponding to a point mass. Later, we followed the idea of Dyakonov and Khavinson to construct in a similar way a non-zero function in $\mathcal{K}_\theta \cap \mathcal{A}^\infty$ for any singular inner functions θ corresponding to a singular measure which assigns positive mass to some Carleson set of zero Lebesgue measure. In our approach, we exploited that $\overline{zm}\theta \in C^\infty(\mathbb{T})$ for analytic m with appropriate zero set, and that the projection operator P_+ preserves smoothness. For a general θ, following naively the above idea leads to $\overline{zm}\theta \in C(\mathbb{T})$ for any non-zero analytic m which vanishes at the discontinuities of θ on $\mathbb{T}$. However, it is known that P_+ does not typically preserve continuity, so we cannot conclude that $P_+\overline{zm}\theta \in K_\theta \cap \mathcal{A}$.

Problem 4.1. *Given a singular inner function θ, construct explicitly a non-zero function in $K_\theta \cap \mathcal{A}$.*

The author is not aware of any such constructions which work in the general case, even though the problem has been previously highlighted in several works (see, for instance, the discussion in [10]).

4.2. **Constructive proof of Khrushchev's theorem.** Khrushchev's Theorem 3.5 and Theorem 3.6 deal with the question of existence of a function $k \in L^2(\mathbb{T})$ supported on a given set E for which the projection P_+k lies in the class $\mathcal{A}$ or $\mathcal{A}^\infty$. Both proofs initially given by Khrushchev are non-constructive. In the latter case, the necessary and sufficient condition for existence of non-zero such k is that E contains a Carleson set of positive Lebesgue measure, and under this condition, an explicit construction of such k has been given in [29]. The construction is, again, based on the smoothness preservation property of P_+.

A constructive proof of Theorem 3.5 is not known to the author.

Problem 4.2. *Given a set $E \subset \mathbb{T}$ of positive Lebesgue measure, construct explictly a non-zero function k which vanishes outside of E, and satisfies $P_+k \in \mathcal{A} \setminus \{0\}$.*

Through Proposition 2.2, a solution of the above problem would provide us with an explicit construction of a non-zero function in $\mathcal{H}(b) \cap \mathcal{A}$. Note that k in Problem 4.2 must necessarily be a function of rather complicated structure. For instance, in the case that E contains no intervals, the requirement for k to live only on E implies that k must be highly discontinuous.

4.3. **The remaining Gevrey classes.** With the exception of the Gevrey classes $\mathcal{G}_\beta$ corresponding to the parameter range $\beta \in (0, 1/2)$, all other regularity classes introduced in Section 1 have appeared in our discussion. We have seen that $K_\theta \cap \mathcal{G}_{1/2} = \{0\}$ whenever θ is singular inner, and this result is sharp in the sense that for every $\beta \in (0, 1/2)$, the intersection $K_\theta \cap \mathcal{G}_\beta$ is non-trivial for the singular inner function θ appearing in Example 3.1. One way to see this is to apply Lemma 3.9 and Remark 3.10 to the space $X^* = \mathcal{P}^2(\mathcal{E}_{\widetilde{\beta},c}dA)$ for $\widetilde{\beta} \in (0, 1)$ which we defined in Section 2.3, and argue that θ is not cyclic in this space. This latter fact is not immediate, but can be proved for instance by the use of techniques of Limani introduced in the artice [27], which deals with certain extensions of the Korenblum-Roberts cyclicity theory.

Fixing $\beta \in (0, 1/2)$ we have that $\widetilde{\beta} = \frac{\beta}{1-\beta} \in (0, 1)$. Just like the regularity class $\mathcal{A}^\infty$ corresponds to the Carleson condition in (3.4) through Theorem 3.3 and Theorem 3.8, the class $\mathcal{G}_\beta$ appears to be related to a generalized Carleson-type condition

$$\sum_{\ell} |\ell|^{1-\widetilde{\beta}} < \infty, \tag{4.1}$$

where, as before, $\{\ell\}$ is the system of maximal open arcs complementary to a closed set $E \subset \mathbb{T}$. If E satisfies (4.1) with $1 - \widetilde{\beta} = \alpha$, then E is sometimes called a *α-Carleson set*. Results of Khrushchev from [22], the splitting sequence constructions from [7], and an argument similar to the one used in the proof of Theorem 3.8, shows that in the special case that b is outer, the intersection $\mathcal{H}(b) \cap \mathcal{G}_\beta$, $\beta \in (0, 1/2)$, is non-trivial if and only if there exists a $(1 - \widetilde{\beta})$-Carleson set of positive Lebesgue measure on which $\log \Delta_b$ is integrable. To complete a version of Theorem 3.8 corresponding to the class $\mathcal{G}_\beta$, we would need a corresponding singular inner function cyclicity theorem.

Problem 4.3. *Find out if an extension of Korenblum's cyclicity theorem to the context of $(1-\widetilde{\beta})$-Carleson sets is valid. Namely, decide if a singular inner function $\theta = S_\nu$ is cyclic in $\mathcal{P}^2(\mathcal{E}_{\widetilde{\beta},c} dA)$ if and only if $\nu(E) = 0$ for every $(1 - \widetilde{\beta})$-Carleson set of zero Lebesgue measure.*

Was the above statement verified, then through (2.7), (2.9) and Lemma 3.9 we would deduce a version of the Dyakonov-Khavinson Theorem 3.3 for the Gevrey classes. Together with the above remark, we would deduce a $\mathcal{G}_\beta$-version of Theorem 3.8.

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