On model spaces and density of functions regular on the boundary

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Abstract

We characterize the model spaces K_Θ in which functions with smooth boundary extensions are dense. It turns out that such approximation is possible if and only if the singular measure associated to the singular inner factor of Θ is concentrated on a countable union of Beurling-Carleson sets. In particular, if there exists a restriction of the associated singular measure which does not assign positive measure to Beurling-Carleson sets, then we prove that even spaces of functions of lower order regularity, such as the Hölder classes and large collections of analytic Sobolev spaces, fail to be dense. In contrast to earlier results on density of functions with continuous extensions to the boundary in K_Θ and related spaces, our method is not based on duality arguments, and is largely constructive.

1 Introduction and main results

A special feature of the spaces of analytic functions which appear in several operator model theories is that it is often difficult to obtain any explicit example of a non-trivial function that is contained in such a space and has boundary values of some degree of regularity, be it continuous, differentiable or smooth. This principle certainly applies to the classical *model spaces* K_Θ, where the identifiable elements of the space are often limited to the reproducing kernel functions. However, being subspaces of the Hardy space, a function in K_Θ behaves at least somewhat well on the boundary, in the sense that it admits an almost everywhere defined boundary function. Our goal here is to carry out the investigation of how well such a boundary function can behave for a dense subset of the model space, and more precisely we will focus on differentiable boundary functions.

We will work in the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$, and our boundary functions will thus be living on the unit circle $\mathbb{T}$. We assume the readers familiarity with the usual facts regarding the Hardy spaces H^p on $\mathbb{D}$ and the boundary behaviours of functions in these spaces (all of it can for instance be found in [9]). In this setting, a model space is constructed in the following way. Let ν be a finite positive singular Borel measure on $\mathbb{T}$ and construct the associated singular inner function

$$S_\nu(z) = \exp\Big(- \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\nu(\zeta)\Big). \tag{1}$$

Pick also an arbitrary convergent Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z},$$

and set $\Theta := BS_\nu$. The space $\Theta H^2 = \{\Theta f : f \in H^2\}$ is then a closed subspace of H^2, and the model space is defined as its orthogonal complement:

$$K_\Theta = H^2 \ominus \Theta H^2.$$

The reproducing kernel of the space is given by

$$k_\Theta(\lambda, z) = \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \overline{\lambda} z}.$$

It is clear from this expression that the kernel functions inherit the boundary behaviour of the inner function Θ, which can in general be very irregular.

During his investigation of the boundary behaviour of model space functions on very fine sets, Aleksandrov established in [1] the density of the intersection $\mathcal{A} \cap K_\Theta$ in K_Θ, where $\mathcal{A}$ denotes the disk algebra, the algebra of analytic functions in $\mathbb{D}$ with continuous extensions to the boundary. It has been later found in [2] and [3] that Aleksandrov's result admits a generalization of to a wider class of spaces, with proofs using non-constructive approaches based on duality. A natural question is if approximation by functions extending continuously to the boundary is close to the best that one can possibly obtain for a general model space. Similarly, one can ask: what conditions need to be met in order to allow approximation by some more regular class of functions?

Let $E \subset \mathbb{T}$ be a closed set of Lebesgue measure zero and consider the complementary open set $\mathbb{T} \setminus E = U$, which can be written as a countable union $U = \cup_k I_k$ of disjoint maximal open circular arcs I_k. Let $|I_k|$ denote the length of the arc I_k. The set E is a *Beurling-Carleson* set if the following entropy condition is satisfied:

$$\sum_k |I_k| \log\left(\frac{1}{|I_k|}\right) < \infty. \tag{2}$$

Beurling-Carleson sets are characterized as zero sets on $\mathbb{T}$ of analytic functions on $\mathbb{D}$ with very regular extensions to the boundary. For instance, Carleson proved in [6] that for any closed set $E \subset \mathbb{T}$ of Lebesgue measure zero satisfying (2) and any positive integer n, there exists an analytic function with zero set E such that all its derivatives up to order n extend continuously to the boundary. We will denote by $\mathcal{A}^n$ the class of analytic functions on $\mathbb{D}$ with derivatives of order n extending continuously to the boundary. We also define

$$\mathcal{A}^\infty := \bigcap_{n=1}^{\infty} \mathcal{A}^n.$$

Novinger in [11] and Taylor and Williams in [14] extended Carleson's result and showed that Beurling-Carleson sets are also zero sets of functions in $\mathcal{A}^\infty$. For $0 < p \leq \infty$, we will also need to define the analytic Sobolev spaces $W^{1,p}_a$ on the unit disc $\mathbb{D}$. That is,

$$W^{1,p}_a = \{f \in \mathbf{Hol}(\mathbb{D}) : f, f' \in L^p(\mathbb{D}, dA)\},$$

where dA denotes the normalized area-measure on $\mathbb{D}$. We regard these spaces as closed subspaces of the classical Sobolev spaces on $\mathbb{D}$, equipped with the standard Sobolev metric $\|f\|_{W^{1,p}} := \|f\|_{L^p} + \|f'\|_{L^p}$.

Dyakonov and Khavinson found in [8] that there exists model spaces K_Θ with the property that $W^{1,p}_a \cap K_\Theta = \{0\}$, for any $p > 1$. According to their result, this happens precisely when $\Theta = S_\nu$ and the singular measure ν has the property that $\nu(E) = 0$ for any Beurling-Carleson set E. Conversely, they show that the model space K_Θ contains a non-zero function in $\mathcal{A}^\infty$ if either Θ has a Blaschke factor or if $\nu(E) > 0$ for some Beurling-Carleson set E. With these observations at hand, we shall in Section 2.2 establish a certain decomposition of singular measures. Any finite positive singular Borel measure ν on $\mathbb{T}$ can be expressed as a sum

$$\nu = \nu_{\mathcal{C}} + \nu_{\mathcal{K}} \tag{3}$$

where the two measures are mutually singular, $\nu_{\mathcal{C}}$ is essentially concentrated on Beurling-Carleson sets, in the sense that there exists an increasing sequence of Beurling-Carleson sets $\{E_n\}_{n\geq 1}$ such that $\lim_{n\to\infty} \nu_{\mathcal{C}}(E_n) = \nu_{\mathcal{C}}(\mathbb{T})$, while $\nu_K(E) = 0$ for any Beurling-Carleson set. Measures which do not charge Beurling-Carleson sets appear notably in the work of Korenblum in [10] and Roberts in [12]. For these reasons, in the decomposition in (3), we will refer to the measure $\nu_{\mathcal{C}}$ as the *Beurling-Carleson part*, and the measure ν_K as the *Korenblum-Roberts part*. The main result of our investigation is the following.

Theorem 1.1. *Let $\Theta = BS_\nu$ be an inner function with Blaschke product B and singular inner function S_ν with associated singular measure ν on $\mathbb{T}$. Then the following conditions are equivalent:*

(i) $\mathcal{A}^\infty \cap K_\Theta$ *is dense in* K_Θ,

(ii) $W_a^{1,p} \cap K_\Theta$ *is dense in* K_Θ*, for any* $1 < p \leq \infty$,

(iii) the Korenblum-Roberts part $\nu_{\mathcal{K}}$ *of* ν *is trivial. That is,* $\nu = \nu_{\mathcal{C}}$.

Some remarks are in order. If the Korenblum-Roberts part of the singular measure is non-trivial, then the associated model space contains functions which cannot be approximated even by analytic Sobolev functions $W_a^{1,p} \cap K_\theta$, for any $p > 1$. This means that for a wide range of analytic function spaces $\mathcal{M}$ lying between $W_a^{1,p}$ and $\mathcal{A}^\infty$, the density of $\mathcal{M} \cap K_\Theta$ in K_Θ is equivalent to condition (*iii*) of Theorem 1.1. For instance, the analytic Hölder class Λ_a^α, consisting of analytic functions on $\mathbb{D}$ with Hölder continuous boundary values on $\mathbb{T}$ of order $\alpha \in (0,1)$ is a subset of $W_a^{1,p}$ for all $1 < p < 1/(1-\alpha)$. We summarize these observations as a corollary of Theorem 1.1.

Corollary 1.2. *Let $\Theta = BS_\nu$ be an inner function with Blaschke product B and singular inner function S_ν with associated singular measure ν on $\mathbb{T}$, and let $\mathcal{M}$ be any of the Hölder classes Λ_a^α for $\alpha \in (0,1)$ or $\mathcal{A}^k$ for any integer $k \geq 1$. Then $\mathcal{M} \cap K_\Theta$ is dense in K_Θ if and only if the Korenblum-Roberts part ν_K of ν is trivial.*

Our theorem treats the density of analytic Sobolev spaces $W_a^{1,p}$ for $p > 1$. When $0 < p < 1$ the situation is simple, since then the spaces $W_a^{1,p}$ contain all the bounded analytic functions, and thus $W_a^{1,p} \cap K_\Theta$ is obviously dense in K_Θ, for all inner functions Θ. In the case of $p = 1$, little is known and the problem seems interesting. In the proof of the necessity of condition (iii) in Theorem 1.1, we use a deep result by Korenblum and Roberts from [10] and [12], which asserts that inner functions associated with singular measures which do not assign positive measure to Beurling-Carleson sets are cyclic vectors for the forward shift operator on the Bergman spaces. An extension of our result to the space $W_a^{1,1}$ seems to be related to the open problem of determining if the condition $\nu(E) = 0$ for any Beurling-Carleson set is sufficient for the associated singular inner function S_ν to be a weak* cyclic vector for the forward shift operator on the Bloch space. Similar remarks regarding connections to this open problem also appeared in [8] and we refer the reader to [5] and [4] for further details and progress on this question. Notably, weak* cyclicity has been established for a particular example of a singular inner function in [4], and consequently there exists an inner function

Θ, such that $W^{1,1}_a \cap K_\Theta = \{0\}$. In particular, this means that there are model spaces which do not contain non-trivial functions in the so-called *Wiener algebra*, which consists of analytic functions with absolutely summable Taylor coefficients.

The method we employ to establish the density results is to a large extent constructive. The approach is based on co-analytic Toeplitz operators and smoothing out the boundary singularities of reproducing kernel functions by multiplication with the Toeplitz operator symbol. It seems therefore that the approach can be generalized to other spaces with similar properties on which the co-analytic Toeplitz operators are bounded. One can easily see that our proofs carry over to model subspaces of H^p-spaces, for $p > 1$. These can be defined as the the pre-annihilator of the set θH^q under the usual duality between H^p and H^q, $1/p + 1/q = 1$.

This paper is organized as follows. In the preliminary Section 2 and Section 3 we have gathered the preparatory work for the proof of Theorem 1.1. Section 3 contains the main technical argument, which is the construction of a certain sequence of smoothing functions which converge to the identity. The last section is devoted to the proof of our main result and is divided into two subsections. The first part contains the proof of the necessity, which refers to proving that (ii) implies (iii) in Theorem 1.1, while the second part contains the proof of the sufficiency, which refers to deducing (i) from (iii).

2 Preliminaries

2.1 Riesz and Herglotz transforms Let $P_+ : L^2(\mathbb{T}) \to H^2$ denote the *Riesz projection* given by

$$P_+ f(z) := \int_{\mathbb{T}} \frac{\zeta}{\zeta - z} f(\zeta) dm(\zeta), \qquad z \in \mathbb{D},$$

and the *Herglotz transform* by

$$Hf(z) := \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} f(\zeta) dm(\zeta), \qquad z \in \mathbb{D},$$

where dm denotes the normalized Lebesgue measure on $\mathbb{T}$. With simple algebra we can express the Herglotz transform in terms of the Riesz projection by $Hf = 2P_+ f - P_+ f(0)$. The following results on how these operators preserve regularity of functions on the boundary $\mathbb{T}$ are well-known and can for instance be deduced from similar properties of the so-called conjugate operator, which is treated carefully in Chapter III of [9]. In what follows we shall denote by $C^n(\mathbb{T})$ the class of n times continuously differentiable functions on $\mathbb{T}$ and set $C^\infty(\mathbb{T}) = \cap_n C^n(\mathbb{T})$.

Lemma 2.1. *The Riesz projection P_+ and the Herglotz transform H map $C^{n+1}(\mathbb{T})$ into $\mathcal{A}^n$. In particular, they map $C^\infty(\mathbb{T})$ into $\mathcal{A}^\infty$.*

2.2 A decomposition of singular measures We establish the abstract decomposition result for singular measures which was stated in the introduction. The result is certainly not new and a version of it has been briefly mentioned in [12]. We provide a short proof because the result plays an important role in our investigation.

Proposition 2.2. *Let ν be a positive finite singular Borel measure on $\mathbb{T}$. Then ν decomposes uniquely as a sum of mutually singular measures*

$$\nu = \nu_{\mathcal{C}} + \nu_{\mathcal{K}}, \tag{4}$$

such that there exists a sequence of Beurling-Carleson sets $\{E_n\}_{n\geq 1}$ satisfying

$$\lim_{n\to\infty} \nu_{\mathcal{C}}(E_n) = \nu_{\mathcal{C}}(\mathbb{T})$$

and $\nu_{\mathcal{K}}$ vanishes on every Beurling-Carleson set.

Proof. Consider the quantity

$$\sup\{\nu(E) : E \subset \mathbb{T} \text{ Beurling-Carleson set }\}. \tag{5}$$

Since the Beurling-Carleson condition in (2) is preserved under finite unions, we can pick an increasing sequence of Beurling-Carleson sets $\{E_n\}_{n\geq 1}$, such that $\nu(E_n)$ increases up to the supremum in (5), and set $F = \cup_{n\geq 1} E_n$. We now decompose ν according to

$$\nu = \nu_{\mathcal{K}} + \nu_{\mathcal{C}}, \tag{6}$$

where $\nu_{\mathcal{C}} := \nu|_F$ and $\nu_{\mathcal{K}} := \nu|_{\mathbb{T}\setminus F}$. By construction $\nu_{\mathcal{K}}$ and $\nu_{\mathcal{C}}$ are mutually singular, and it readily follows that $\lim_{n\to\infty} \nu_{\mathcal{C}}(E_n) = \nu_{\mathcal{C}}(\mathbb{T})$ and that $\nu_{\mathcal{K}}(E) = 0$ for any Beurling-Carleson set $E \subset \mathbb{T}$. In order to prove uniqueness of the decomposition in (6), it is enough to note that the measure $\nu_{\mathcal{C}}$ is uniquely determined by

$$\nu_{\mathcal{C}}(A) = \sup\{\nu(A \cap E) : E \subset \mathbb{T} \text{ Beurling-Carleson set}\}$$

for any Borel set $A \subseteq \mathbb{T}$.

$\square$

2.3 Functionals on analytic Sobolev spaces Let $1 < p < \infty$ and denote the classical Bergman spaces by L^p_a. That is, the closed subspace of $L^p(\mathbb{D}, dA)$ consisting of analytic functions on $\mathbb{D}$. Also recall the analytic Sobolev spaces $W^{1,p}_a$, previously defined in the introduction. In this section, we shall demonstrate that the Cauchy H^2-pairing with L^q_a-functions induce bounded linear functionals on $W^{1,p}_a$, where $q = p/(p-1)$. That is, for every $g \in L^q_a$ and $f \in W^{1,p}_a$, the following limit exists

$$\lim_{r \to 1-} \int_{\mathbb{T}} f(r\zeta)\overline{g(r\zeta)} dm(\zeta). \tag{7}$$

Moreover, there exists a constant $C_p > 0$, only depending on p, such that

$$\sup_{0<r<1} \left| \int_{\mathbb{T}} f(r\zeta)\overline{g(r\zeta)} dm(\zeta) \right| \leq C_p \|f\|_{W^{1,p}} \|g\|_{L^q}. \tag{8}$$

Lemma 2.3. *Let $1 < p < \infty$ and $q = p/(p-1)$. Then for every $g \in L^q_a$ the Cauchy H^2-pairing in (7) exists and the estimate in (8) holds.*

Proof. Observe that it suffices to establish the inequality in (8), from which the existence of the limit (7) is easily deduced by means of applying the estimate to a difference of dilations. To this end, we may without loss of generality fix two arbitrary functions $g \in L^q_a \cap H^\infty$ and $f \in W^{1,p}_a \cap H^\infty$. Using Taylor series expansions of f and g, we can express the H^2-pairing in terms of a weighted disc integral via

$$\int_{\mathbb{T}} f(\zeta)\overline{g(\zeta)} dm(\zeta) = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{\left(g'(z) + \frac{g(z) - g(0)}{z} \right)} \left(1 - |z|^2\right) dA. \tag{9}$$

For the sake of abbreviation, let $(Lg)(z) = \frac{g(z)-g(0)}{z}$ denote the backward shift operator applied to g. In what follows, we shall also denote by $C_p > 0$ a constant only depending on $p > 1$, which may change from line to line. Now the integral on the right-hand side of (9) naturally splits into the two terms. We proceed by estimating the first term by Hölder's inequality, which gives

$$\left| \int_{\mathbb{D}} f'(z)\overline{g'(z)} \left(1 - |z|^2\right) dA \right| \leq \|f'\|_{L^p} \left(\int_{\mathbb{D}} \left(1 - |z|^2\right)^q |g'(z)|^q dA \right)^{1/q}.$$

Recall that the rightmost factor in the previous equation defines an equivalent norm (modulo constants) on the Bergman space L^q_a , for $q > 1$ (see for instance [7]). This implies that

$$\left| \int_{\mathbb{D}} f'(z)\overline{g'(z)} \left(1 - |z|^2\right) dA \right| \leq C_p \|f'\|_{L^p} \|g\|_{L^q}.$$

In order to estimate the second term,we similarly estimate the second term by Hölder's inequality, but this time pair the integrands according to

$$\left| \int_{\mathbb{D}} f'(z)\overline{(Lg)(z)}(1-|z|^2)dA \right| \leq \left(\int_{\mathbb{D}} (1-|z|^2)^p |f'(z)|^p dA \right)^{1/p} \|Lg\|_{L^q}$$

Again, since the left factor on the righthand side is equivalent to the L^p_a-norm of f (modulo constants), and the backward shift maps $L : L^q_a \to L^q_a$, we obtain

$$\left| \int_{\mathbb{D}} f'(z)\overline{(Lg)(z)}(1-|z|^2)dA \right| \leq C_p \|f\|_{L^p} \|g\|_{L^q}.$$

Adding up the pieces, we conclude the estimate in (8) and thus establishing the proof of this lemma. □

3 A smoothing approximation to the identity

We will make use of a construction, initially due to Carleson in [6] and later refined by Taylor and Williams in [14]. Let E be a Beurling-Carleson set on $\mathbb{T}$, with complementary open set U that decomposes into $U = \cup_k I_k$, where I_k are disjoint maximal circular arcs of $\mathbb{T}$. Without loss of generality, we may assume that none of the arcs include the point $1 \in \mathbb{T}$. Then, if I_k is extending from $\alpha_k = e^{ia_k}$ to $\beta_k = e^{ib_k}$, we will have $0 \leq a_k < b_k < 2\pi$ and because E is a Beurling-Carleson set, it must be so that

$$\sum_k (b_k - a_k) \log \left(\frac{1}{b_k - a_k} \right) < \infty.$$

Taylor and Williams use this assumption to construct a function $h(t)$ defined for $t \in [0, 2\pi]$, with the properties that $h(t) = \infty$ on the complement of the union of the intervals (a_k, b_k), on those intervals h is of class C^∞ and

$$\lim_{t \to a_k^+} h(t) = \lim_{t \to b_k^-} h(t) = \infty,$$

in such a way that $h(t)$ is integrable on $[0, 2\pi)$ and the outer function

$$g(z) = \exp \left(-\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} h(t) dt \right) \tag{10}$$

belongs $\mathcal{A}^\infty$ with zero set precisely equal to E. Moreover, and this is crucial for our further purposes, the construction of Taylor and Williams is so that

$$E \subseteq \bigcap_{n=1}^{\infty} \{\zeta \in \mathbb{T} : g^{(n)}(\zeta) = 0\}. \tag{11}$$

Further details of their construction can be found in [14, Theorem 3.3].

Versions of our next lemma have appeared in [13] and [14]. We include a proof for the reader's convenience.

Lemma 3.1. *Let ν be a positive measure supported on a Beurling-Carleson set E and S_ν denote the corresponding singular inner function. If g is the function defined above, satisfying* (10) *and* (11)*, then the boundary function of $S_\nu \overline{g}$ is of class $C^\infty(\mathbb{T})$.*

Proof. Let $z, w \in \mathbb{D}$. For any positive integer n we can use the Taylor expansion of order n to write

$$g(z) = \sum_{k=0}^{n-1} g^{(k)}(w) \frac{(z-w)^k}{k!} + \frac{(z-w)^n}{(n-1)!} \int_0^1 g^{(n)}\left(z - \tau(z-w)\right) \tau^{n-1} d\tau. \quad (12)$$

Since g and all of its derivatives extend continuously to the boundary, we see that the Taylor expansion continues to hold for z, w on $\mathbb{T}$. Let $\delta_E(z)$ denote the distance from a point $z \in \overline{\mathbb{D}}$ to the set E. If we pick a point $w \in E$ which is the closest to z, that is $|z-w| = \delta_E(z)$, then, since all derivatives of g vanish at w, only the integral term remains in (12). This shows that

$$|g(z)| \leq \sup_{\zeta \in \mathbb{D}} \left|g^{(n)}(\zeta)\right| \frac{|z-w|^n}{n!} = C_n\, \delta_E(z)^n, \quad (13)$$

where the constant $C_n > 0$, depends on n (and g), but the estimate holds for any positive integer n. Note also that according to (11), the same estimate can be obtained for all derivatives of g. The inner function S_ν has the integral representation in (1) and its derivative is expressed by

$$S_\nu'(z) = -S_\nu(z) \int_{\mathbb{T}} \frac{2\zeta}{(\zeta - z)^2} d\nu(\zeta). \quad (14)$$

This expression does not only hold for $z \in \mathbb{D}$, but also on the complementary open set $U = \mathbb{T} \setminus E$, where S_ν and its derivatives extend analytically, and we have the crude estimate $|S_\nu'(z)| \leq D\delta_E(z)^{-2}$. Likewise, by repeated differentiation we find that

$$|S_\nu^{(k)}(z)| \leq D_k\, \delta_E(z)^{-2k}. \quad (15)$$

Thus, for any positive integer k and l, we deduce from estimates similar to (13) and (15) that

$$\lim_{z \to w \in E} |S_\nu^{(k)}(z) \overline{g^{(l)}(z)}| = 0.$$

The derivatives of $S_\nu \overline{g}$ are linear combinations of terms of the form $S_\nu^{(k)} \overline{g^{(l)}}$. These terms are continuous on U, and the estimates show that they attain values 0 continuously on the set E. We conclude that $S_\nu \overline{g}$ is $C^\infty(\mathbb{T})$. $\square$

We will now modify the construction of the function g above to obtain a sequence of analytic functions which smooths out singularities of inner functions and converges to the identity.

Proposition 3.2. *Let ν be a positive measure supported on a Beurling-Carleson set E, and let S_ν denote the corresponding singular inner function. Then there exists a sequence of analytic functions $\{H_n\}_n$ in $\mathcal{A}^\infty$ with following properties:*

(i) *the boundary function of $S_\nu \overline{H_n}$ is of class $C^\infty(\mathbb{T})$,*

(ii) $\lim_{n\to\infty} H_n(\zeta) = 1$ *holds for Lebesgue almost every $\zeta \in \mathbb{T}$,*

(iii) $\sup_n \|H_n\|_\infty < \infty$.

Proof. Let ψ_n be an increasing sequence of C^∞-functions with $0 \leq \psi_n \leq 1$, compactly supported in the interval $(0,1)$, and such that $\psi_n(t) = 1$ for all $t \in [1/n, 1-1/n]$. Let $U = \cup_{k=1}^\infty (a_k, b_k)$ be as defined in the beginning of the section and consider the functions

$$\phi_n(t) = \sum_{k=1}^n \psi_n\left((t-a_k)/(b_k-a_k)\right).$$

Certainly ϕ_n is C^∞ and $\lim_{n\to\infty} \phi_n(t) = 1$ almost everywhere with respect to the Lebesgue measure. Note that ϕ_n is identically equal to zero in a neighbourhood of any of the endpoints of the intervals appearing in the decomposition of U. Let the function h and corresponding analytic function g given by (10) be as in the construction by Williams and Taylor described above. It follows from the construction that $h\phi_n$ is a C^∞-function, and that $h(1-\phi_n)$ converges pointwise to zero almost everywhere, which by dominated convergence theorem applied to $|h(1-\phi_n)| \leq |h|$ implies that

$$\lim_{n\to\infty} \int_0^{2\pi} h(t)(1-\phi_n(t))dt = 0. \tag{16}$$

We set

$$H_n(z) = \exp\left(-\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} h(t)(1-\phi_n(t))dt \right).$$

Note that certainly H_n are uniformly bounded, so (iii) is satisfied. By construction, any point $\zeta = e^{it}$ with $t \in U$ will have an open neighbourhood around it on which $h(1-\phi_n)$ vanishes for sufficiently large n. Thus for sufficiently large n the functions H_n will have analytic continuations to a neighbourhood of any fixed $\zeta = e^{it}$ with $t \in U$. By (16) we deduce that $\lim_{n\to\infty} H_n(\zeta) = 1$ for each such ζ. Thus $\lim_{n\to\infty} H_n(\zeta) = 1$ for almost every $\zeta \in \mathbb{T}$, which is the assertion in (ii). We can write $H_n = gF_n$, where F_n is the exponential of the Herglotz transform of $h\phi_n$. Since $h\phi_n$ belongs to $C^\infty(\mathbb{T})$, we have that F_n has a boundary function in $C^\infty(\mathbb{T})$ by Lemma 2.1. According to Lemma 3.1, we now conclude that $S_\nu \overline{H_n} = S_\nu \overline{gF_n}$ is of class $C^\infty(\mathbb{T})$, which proves the assertion in (i). □

4 Proof of the main result

4.1 Proof of the necessity

Proof of (ii) ⇒ (iii) of Theorem 1.1. Let $\Theta = BS_\nu$ be an inner function such that (iii) fails, that is, the Korenblum-Roberts part $\nu_{\mathcal{K}}$ is not identically zero. Now factorize $\Theta = \Theta_{\mathcal{C}} S_{\mathcal{K}}$, where $S_{\mathcal{K}} = S_{\nu_{\mathcal{K}}}$ and $\Theta_{\mathcal{C}} = BS_{\nu_{\mathcal{C}}}$ and observe that a simple computation involving reproducing kernels establishes the orthogonal decomposition

$$K_\Theta = K_{\Theta_{\mathcal{C}}} \oplus \Theta_{\mathcal{C}} K_{S_{\mathcal{K}}}.$$

Our aim is to prove that no non-zero function belonging to the subspace $\Theta_{\mathcal{C}} K_{S_{\mathcal{K}}}$ can be approximated by $W^{1,p}_a \cap K_\Theta$, for any $p > 1$. To this end, let $p > 1$ be arbitrary and fixed. Since $\nu_{\mathcal{K}}(E) = 0$ for every Beurling-Carleson set $E \subset \mathbb{T}$, the Korenblum-Roberts Theorem (see for instance Theorem 1 in [10] or Theorem 2 in [12]) implies that the singular inner function $S_{\mathcal{K}}$ is a cyclic vector on the Bergman spaces L^q_a, for any $q > 1$. That is, the subspace $S_{\mathcal{K}} L^q_a$ is dense in L^q_a. Now let $f \in H^\infty \cap K_{S_{\mathcal{K}}} \subset L^q_a$ and pick a sequence of analytic polynomials $\{p_n\}_{n\geq 1}$ such that $S_{\mathcal{K}} p_n \to f$ in L^q_a, as $n \to \infty$. Since multiplication with the bounded analytic function $\Theta_{\mathcal{C}}$ is a continuous operation on L^q_a, we have that $\Theta p_n \to \Theta_{\mathcal{C}} f$ in L^q_a. Observe that the limit function $\Theta_{\mathcal{C}} f$ belongs to K_Θ, since $\Theta_{\mathcal{C}} K_{S_{\mathcal{K}}} \subset K_\Theta$. According to Lemma 2.3, the Cauchy H^2-pairing with an L^q_a-function induces a bounded linear functional on $W^{1,p}_a$, where $p > 1$ satisfies $q = p/(p-1)$. In particular, this implies that for any $g \in W^{1,p}_a \cap K_\Theta$ we have the following

$$\int_{\mathbb{T}} \Theta_{\mathcal{C}}(\zeta) f(\zeta) \overline{g(\zeta)} dm(\zeta) = \lim_{n\to\infty} \int_{\mathbb{T}} \Theta(\zeta) p_n(\zeta) \overline{g(\zeta)} dm(\zeta) = 0.$$

The last equality follows from the fact that g is orthogonal to Θp_n, for each $n \geq 1$. This shows that $\Theta_{\mathcal{C}} f \in (W^{1,p}_a \cap K_\Theta)^\perp$, hence $\Theta_{\mathcal{C}} K_{S_{\mathcal{K}}} \subseteq (W^{1,p}_a \cap K_\Theta)^\perp$. This obviously implies that $W^{1,p}_a \cap K_\Theta$ is not dense in K_Θ. □

4.2 Proof of the sufficiency

Proof of (iii) ⇒ (i) of Theorem 1.1. Let $\Theta = BS_\nu$, where B is a Blaschke product, and S_ν is a singular inner function with corresponding singular measure ν. Suppose that the Korenblum-Roberts part $\nu_{\mathcal{K}}$ vanishes. Then it follows from Proposition 2.2 that there exists a sequence of Beurling-Carleson sets $\{E_n\}_n$, such that $\nu(E_n) \to \nu(\mathbb{T})$ as $n \to \infty$.

For the moment, let us suppose that B is a finite Blaschke product and that ν is supported on a single Beurling-Carleson set. By Proposition 3.2, there exists a sequence $\{H_n\}_n$, such that $\Theta\overline{H_n} \in C^\infty(\mathbb{T})$. The function

$$k_\Theta(\lambda, z) = \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \bar{\lambda}z} \tag{17}$$

is the reproducing kernel of K_Θ, and it is seen from the above formula that for fixed $\lambda \in \mathbb{D}$ the boundary function of $k_\Theta\overline{H_n}$ is of class $C^\infty(\mathbb{T})$. Thus $P_+\left(\overline{H_n}k_\Theta\right)$ belongs to $\mathcal{A}^\infty$ by Lemma 2.1. Note that $P_+\left(\overline{H_n}k_\Theta\right) = T_{\overline{H_n}}(k_\Theta)$, where $T_{\overline{H_n}}$ denotes the Toeplitz operator with the bounded co-analytic symbol $\overline{H_n}$, and recall that K_Θ is an invariant subspace for such operators. Thus $T_{\overline{H_n}}(k_\Theta) \in \mathcal{A}^\infty \cap K_\Theta$. Moreover, since H_n tends to 1 almost everywhere on $\mathbb{T}$ and $\{H_n\}_n$ is uniformly bounded, we see easily that the Toeplitz operators $\{T_{\overline{H_n}}\}_n$ converge to the identity operator in the strong operator topology. Consequently, $T_{\overline{H_n}}(k_\Theta) \to k_\Theta$ in H^2, as $n \to \infty$. Combining this with the fact that finite linear combinations of reproducing kernels are dense in K_Θ we obtain the density of $\mathcal{A}^\infty \cap K_\Theta$ in K_Θ, for this special form of Θ.

We now treat the general case. Let $\Theta_N = B_N S_{\nu_N}$ be the inner function where B_N consists of the product of the first N Blaschke factors in the Blaschke product B, and where S_{ν_N} is the singular inner function corresponding to the measure ν_N which is the restriction of ν to the Beurling-Carleson set E_N. Since $\nu = \nu_N + \mu_N$, where μ_N is a positive measure, we easily see that S_{ν_N} is a factor of S_ν, and therefore that Θ_N is a factor in Θ. It then follows that $\Theta H^2 \subset \Theta_N H^2$, and therefore we have the containment $K_{\Theta_N} \subset K_\Theta$. By our assumption, the measures ν_N converge in the total variation norm to ν, which implies that S_{ν_N} converge uniformly on compact subsets of $\mathbb{D}$ to S_ν. Similarly, B_N converges uniformly on compacts to B, and consequently $\Theta_N \to \Theta$, uniformly on compacts. Since Θ_N is also a uniformly bounded sequence, we have that Θ_N converges weakly to Θ in H^2. This, in view of the reproducing kernel formula (17), makes it clear that $k_{\Theta_N}(\lambda, z) \to k_\Theta(\lambda, z)$ weakly in H^2, for any fixed $\lambda \in \mathbb{D}$. By the Banach-Saks Theorem, $k_\Theta(\lambda, z)$ lies in the H^2-norm closure of the convex hull of the functions $\{k_{\Theta_N}(\lambda, z)\}_N$. Each of the kernels $k_{\Theta_N}(\lambda, z)$ can be approximated to arbitrary precision by a function in $\mathcal{A}^\infty \cap K_{\Theta_N} \subset \mathcal{A}^\infty \cap K_\Theta$, and therefore it follows that $k_\Theta(\lambda, z)$

can also be approximated to arbitrary precision by functions in $\mathcal{A}^\infty \cap K_\Theta$. This establishes that $\mathcal{A}^\infty \cap K_\Theta$ is dense in K_Θ. $\square$

References

[1] A. B. Aleksandrov, *Invariant subspaces of shift operators. An axiomatic approach*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 113 (1981), pp. 7–26, 264.

[2] A. Aleman and B. Malman, *Density of disk algebra functions in de Branges–Rovnyak spaces*, Comptes Rendus Mathematique, 355 (2017), pp. 871–875.

[3] ——, *Hilbert spaces of analytic functions with a contractive backward shift*, Journal of Functional Analysis, 277 (2019), pp. 157–199.

[4] J. Anderson, J. Fernandez, and A. L. Shields, *Inner functions and cyclic vectors in the Bloch space*, Transactions of the American Mathematical Society, 323 (1991), pp. 429–448.

[5] L. Brown and A. L. Shields, *Multipliers and cyclic vectors in the Bloch space.*, The Michigan Mathematical Journal, 38 (1991), pp. 141–146.

[6] L. Carleson, *Sets of uniqueness for functions regular in the unit circle*, Acta mathematica, 87 (1952), pp. 325–345.

[7] P. Duren and A. Schuster, *Bergman spaces*, vol. 100 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2004.

[8] K. Dyakonov and D. Khavinson, *Smooth functions in star-invariant subspaces*, Contemporary Mathematics, 393 (2006), p. 59.

[9] J. Garnett, *Bounded analytic functions*, vol. 236, Springer Science & Business Media, 2007.

[10] B. Korenblum, *Cyclic elements in some spaces of analytic functions*, Bulletin of the American Mathematical Society, 5 (1981), pp. 317–318.

[11] W. P. Novinger, *Holomorphic functions with infinitely differentiable boundary values*, Illinois Journal of Mathematics, 15 (1971), pp. 80–90.

[12] J. Roberts, *Cyclic inner functions in the Bergman spaces and weak outer functions in H^p, $0 < p < 1$*, Illinois Journal of Mathematics, 29 (1985), pp. 25–38.

[13] H. Shapiro, *Some remarks on weighted polynomial approximations of holomorphic functions*, Mathematics of the USSR-Sbornik, 2 (1967), p. 285.

[14] B. A. Taylor and D. L. Williams, *Ideals in rings of analytic functions with smooth boundary values*, Canadian Journal of Mathematics, 22 (1970), pp. 1266–1283.

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