

On the problem of smooth approximations in $\mathcal{H}(b)$ and connections to subnormal operators

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Abstract

For the class of de Branges-Rovnyak spaces $\mathcal{H}(b)$ of the unit disk \mathbb{D} defined by extreme points b of the unit ball of H^∞ , we study the problem of approximation of a general function in $\mathcal{H}(b)$ by a function with an extension to the unit circle \mathbb{T} of some degree of smoothness, for instance satisfying Hölder estimates or being differentiable. We will exhibit connections between this question and the theory of subnormal operators and, in particular, we will tie the possibility of smooth approximations to properties of invariant subspaces of a certain subnormal operator. This leads us to several computable conditions on b which are necessary for such approximations to be possible. For a large class of extreme points b we use our result to obtain explicit necessary and sufficient conditions on the symbol b which guarantee the density of functions with differentiable boundary values in the space $\mathcal{H}(b)$. These conditions include an interplay between the modulus of b on \mathbb{T} and the spectrum of its inner factor.

1 INTRODUCTION AND THE MAIN RESULT

Let $b : \mathbb{D} \rightarrow \mathbb{D}$ be an analytic function, where \mathbb{D} denotes the unit disk of the complex plane \mathbb{C} . The space $\mathcal{H}(b)$, introduced in [4] by de Branges and Rovnyak is the Hilbert space of analytic functions on \mathbb{D} with reproducing kernels of the form

$$k_b(\lambda, z) = \frac{1 - \overline{b(\lambda)}b(z)}{1 - \bar{\lambda}z}, \quad \lambda, z \in \mathbb{D}. \quad (1)$$

De Branges and Rovnyak used $\mathcal{H}(b)$ and vector-valued generalizations of these spaces, together with the classical backward shift operator, as models for contractive operators on a Hilbert space. The general theory of $\mathcal{H}(b)$ was further developed by Sarason in a series of articles, and he summarized the work of himself and other researchers in the book [18] (see also the more recent monograph [5], [6]). Importantly, the work of Sarason exhibited several connections between the theory of $\mathcal{H}(b)$ -spaces and other parts of operator and function theory.

The purpose of this note is to bring to light another such connection. We will show how the problem of smooth approximations in $\mathcal{H}(b)$ -spaces is connected to the theory of subnormal operators and derive some concrete results from this connection. The approximation problem can be stated in the following way. Let X be some class of analytic functions in \mathbb{D} with extensions to the unit circle \mathbb{T} of some degree of smoothness. What are the conditions on b which guarantee that for each $f \in \mathcal{H}(b)$ and each $\epsilon > 0$ we can find a function $g \in X \cap \mathcal{H}(b)$ such that $\|f - g\|_{\mathcal{H}(b)} < \epsilon$? The problem is only interesting in the case of b being an extreme point of the unit ball, or equivalently, in the case of the boundary values of b being so close to 1 in modulus that $\log(1 - |b|)$ is not integrable on \mathbb{T} . If this quantity is integrable then it is well-known, and it was proved by Sarason, that the space $\mathcal{H}(b)$ contains all polynomials and they form a dense subset of the space (see [18]). In contrast, in the extreme case only very few polynomials can be a member of $\mathcal{H}(b)$, and indeed it is often a non-trivial task to verify the existence of any non-zero function with any degree of boundary regularity in such a space. In this note, we will thus deal exclusively with the extreme case.

Function classes X that we will consider in this note are to be more than simply continuous on \mathbb{T} . The continuous case has been previously investigated in [1], and the conclusion is that no assumptions on b whatsoever are necessary for this kind of approximation: functions in $\mathcal{H}(b)$ which extend continuously to the boundary (we shall denote by \mathcal{A} the space of analytic functions in \mathbb{D} that extend continuously to \mathbb{T}) are always dense in $\mathcal{H}(b)$. In the present work, we want to instead study approximations by functions satisfying, for instance, classical Hölder or Lipschitz estimates, but also smaller classes such as \mathcal{A}^k , which is the space of analytic functions in \mathbb{D} such that f , together with its first k derivatives, extends continuously to \mathbb{T} . A simple and useful way to capture some notion of boundary smoothness of an analytic function in \mathbb{D} is to specify the rate at which the Taylor coefficients of the function decrease. For a positive number $\alpha > 0$, X_α will thus denote the

following Hilbert space of analytic functions in \mathbb{D} :

$$X_\alpha = \{f \in \text{Hol}(\mathbb{D}) : \|f\|_\alpha^2 := \sum_{n=0}^{\infty} (n+1)^\alpha |f_n|^2 < \infty\}, \quad (2)$$

the symbol f_n denoting the n :th Taylor coefficient of the Taylor expansion of f at the origin. The class X_1 is the classical Dirichlet space, and for large α the space X_α consists entirely of functions with a number of derivatives which extend continuously to \mathbb{T} . In fact, when $\alpha = 2k$ with positive integer k , then we basically recover the classical Hardy-Sobolev spaces.

A subnormal operator M is the restriction to an invariant subspace \mathcal{K} of a normal operator on a Hilbert space \mathcal{H} . Thus $\mathcal{K} \subseteq \mathcal{H}$ and $M\mathcal{K} \subseteq \mathcal{K}$. A typical example is $\mathcal{P}^2(\mu) \subseteq L^2(\mu)$ and the operator $M_z : f(z) \mapsto zf(z)$, where μ is a compactly supported measure in the complex plane, $L^2(\mu)$ is the usual space of μ -square-integrable functions, and $\mathcal{P}^2(\mu)$ is the closed subspace spanned by analytic polynomials. Any cyclic subnormal operator appears in this way for some measure μ (see, for instance, [3]). The measures μ which will appear in the context of our approximation problems are of the form $d\mu = w dA + \omega dm$, where dA and dm are (normalized) area and arclength measures on the disk \mathbb{D} and circle \mathbb{T} , respectively, and w and ω are integrable functions, w being radial: $w(z) = w(|z|)$. In this case, the measure is supported on the closed unit disk and the operator M_z is clearly a contraction.

The following problem for measures of the above form has already been extensively studied in multiple contexts, sometimes in disguise: when is $M_z : \mathcal{P}^2(\mu) \rightarrow \mathcal{P}^2(\mu)$ completely non-unitary? By this we mean that there exists no M_z -invariant subspace of $\mathcal{P}^2(\mu)$ on which M_z acts as a unitary operator. In the setting which we consider below, we will need the slightly stronger assumption that M_z is *completely non-isometric*, that is, admits no M_z -invariant subspace on which this operator acts as an isometry. More explicitly, there should exist no non-zero element $f \in \mathcal{P}^2(\mu)$ which satisfies

$$\|f\|_{\mu,2}^2 := \int_{\mathbb{D}} |f(z)|^2 d\mu(z) = \int_{\mathbb{D}} |z^n f(z)|^2 d\mu(z) = \|M_z^n f\|_{\mu,2},$$

for all positive integers n . The number of works on this problem is huge, and we would like to in particular mention articles [12] and [10] which will provide interesting examples relevant for the present work. If w is, say, continuous and positive on \mathbb{D} (as it will be in our applications), then convergence of a sequence of polynomials $\{p_n\}_{n \geq 1}$ in $\mathcal{P}^2(\mu)$ to f implies that the restriction of

f to \mathbb{D} is an analytic function on \mathbb{D} , and it is easy to see that $M_z : \mathcal{P}^2(\mu) \rightarrow \mathcal{P}^2(\mu)$ is completely non-isometric if and only if $\mathcal{P}^2(\mu)$ does not contain any non-zero element $f \in L^2(\mu)$ which vanishes on \mathbb{D} , and thus lives only on the circle \mathbb{T} . Note that in this case, and this is very important for our purposes, $\mathcal{P}^2(\mu)$ can be interpreted as a space of analytic functions on \mathbb{D} . However, even in the case of measures of the form above, a complete characterization of when this happens seems to be very difficult to obtain. We will see how this problem is connected to smooth approximations in $\mathcal{H}(b)$, and that the many available partial results will nevertheless help us reach several interesting conclusions.

Our main theorem will be stated in terms of a measure of the form mentioned above. Given an extreme point b of the unit ball of H^∞ , and a parameter $\alpha > 0$, we shall now construct a measure $\mu = \mu(b, \alpha)$ on the closed unit disk $\overline{\mathbb{D}}$ and a corresponding space $\mathcal{P}^2(\mu)$, which encodes in properties of the operator $M_z : \mathcal{P}^2(\mu) \rightarrow \mathcal{P}^2(\mu)$ the possibility of approximation of an arbitrary function in $\mathcal{H}(b)$ by a function in X_α . Let

$$w(z) = w_{\alpha-1}(z) := (1 - |z|^2)^{\alpha-1} \quad (3)$$

be the weight on the disk \mathbb{D} ,

$$\Delta(z) := \sqrt{1 - |b(z)|^2} \quad (4)$$

be the weight on \mathbb{T} , and define the measure

$$d\mu = d\mu(b, \alpha) := w_{\alpha-1}dA + \Delta^2 dm. \quad (5)$$

The following is our main theorem on the connection between the smooth approximation problem in $\mathcal{H}(b)$ -spaces and the structure of invariant subspaces of the operator $M_z : \mathcal{P}^2(\mu) \rightarrow \mathcal{P}^2(\mu)$. In the statement below, the space \mathcal{N}^+ is the Smirnov class of the unit disk. Recall that a function f belongs to \mathcal{N}^+ if and only if it can be expressed as $f = u/v$, where u, v are bounded and v is an outer function (the elementary theory of the Smirnov class can be found in [7]). If f is a function in $\mathcal{P}^2(\mu)$, we denote by $[f]$ the smallest closed M_z -invariant subspace containing f . Clearly, $g \in [f]$ if and only if there exists a sequence of polynomials $\{p_n\}_{n \geq 1}$ such that $\lim_{n \rightarrow \infty} \|fp_n - g\|_{\mu, 2} = 0$.

Theorem 1.1. *Let b be an extreme point of the unit ball of H^∞ , $\alpha > 0$ and μ be defined by (5). Further, let θ be the inner factor of b and the space X_α be defined according to (2). The following statements are equivalent.*

- (i) The linear manifold $X_\alpha \cap \mathcal{H}(b)$ is dense in $\mathcal{H}(b)$.
- (ii) The operator $M_z : \mathcal{P}^2(\mu) \rightarrow \mathcal{P}^2(\mu)$ is completely non-isometric and the following property holds: if $f \in \mathcal{N}^+$ is a function contained in the invariant subspace $[\theta] \subset \mathcal{P}^2(\mu)$, then $f/\theta \in \mathcal{N}^+$.

The density of function classes X_α in $\mathcal{H}(b)$ is thus reduced to the study of some specific structure of certain invariant subspaces of $\mathcal{P}^2(\mu)$. The inner function division property described in (ii) should be compared to the arguments appearing in [1], where a similar condition plays a role in establishing the density of $\mathcal{A} \cap \mathcal{H}(b)$ in $\mathcal{H}(b)$. Moreover, it can be deduced from results of [15] that the density of smooth functions $\mathcal{A}^\infty := \bigcap_{n \geq 1} \mathcal{A}^k$ in a model space K_θ (that is, an $\mathcal{H}(b)$ space generated by an inner function $b = \theta$) is equivalent to this division property with $\mu = dA$.

The rest of the paper is structured as follows. In the next section, we present applications of the Theorem 1.1 which will provide us a good understanding of what the different properties of the symbol b imply for our approximation question. In particular, we shall illustrate three principal ways in which smooth approximations are prohibited. We will also construct a very large family of symbols b which avoid in some ways the three mentioned obstructions, and we will show that for this family of symbols b , the set $X_\alpha \cap \mathcal{H}(b)$ is dense in $\mathcal{H}(b)$. In Section 3, we present some basic facts about the X_α spaces. The proof of Theorem 1.1 is deferred to Sections 4, 5 and 6.

2 APPLICATIONS OF THE THEOREM

Theorem 1.1 and the theory of subnormal operators allows for immediate deduction of interesting facts about smooth approximations on $\mathcal{H}(b)$ -spaces, defined by extreme points b of the unit ball of H^∞ . In fact, in parallel with this work, the authors developed the theory in [14] partly for the purpose of this application, and that work will provide us with very general results and interesting examples. Outside this reference, also examples from [10] and [11] will be very useful.

We start by presenting the explicit quantitative conditions that we have found which prohibit smooth approximations in $\mathcal{H}(b)$ -spaces. As mentioned earlier, our investigation exhibits three principal ways in which this happens. The first way deals with the structure of the carrier set of the weight Δ

defined in (4) above, second one deals with the size of the weight Δ on the carrier set, and the third deals with an interplay between the singularities of the inner factor of b and the location of the mass of Δ . For the purpose of our presentation we will need to introduce classes of sets and measures on the circle \mathbb{T} . For a closed set E on \mathbb{T} , we consider the complement $\mathbb{T} \setminus E = \cup_k A_k$, where A_k are disjoint open circular arcs. The set E is a *Beurling-Carleson set* if the following condition is satisfied:

$$\sum_k |A_k| \log \left(\frac{1}{|A_k|} \right) < \infty. \quad (6)$$

These sets appear in the theory of Bergman spaces of Korenblum (see [8]), often with the extra condition that the Lebesgue measure $|E|$ of E is zero, and they appear also in the theory of subnormal operators, where they usually have positive measure (see [12] or [10]). We will use Beurling-Carleson sets of both zero and positive measure in the presentation of our examples. In the examples below, we let μ be given by (5) and denote by

$$E := \{\zeta \in \mathbb{T} : |b| < 1\} = \{\zeta \in \mathbb{T} : \Delta > 0\} \quad (7)$$

the carrier set of Δ . Our first application of Theorem 1.1 shows that the structure of E can prohibit smooth approximations.

Corollary 2.1. *Assume that E is such that there exists no Beurling-Carleson set G of positive measure such that $|G \setminus E| = 0$. Then $X_\alpha \cap \mathcal{H}(b)$ is not dense in $\mathcal{H}(b)$, for any $\alpha > 0$.*

The assumption means that E essentially does not contain Beurling-Carleson sets of positive measure. In particular, E contains no interval, but of course much more is true. The corollary follows from applying [10, Theorem 10.1] to our criterion given in Theorem 1.1. See also [12, Theorem 1.2]. The cited results imply that $L^2(\mu|_{\mathbb{T}})$ is a summand of $\mathcal{P}^2(\mu)$, and on this subspace M_z certainly acts as an isometric operator. Moreover, sets of Cantor-type are constructed in [10] which satisfy the condition in Corollary 2.1 and can be used to produce our first explicit examples of $\mathcal{H}(b)$ in which no smooth approximations are possible. Take such a set E and any b such that $|b| = 1$ on $\mathbb{T} \setminus E$ but $|b| < 1$ almost everywhere on E . Irrespective of the inner factor of b , the linear manifolds $X_\alpha \cap \mathcal{H}(b)$ are never dense

in $\mathcal{H}(b)$ for such choices of b . In fact, it can be deduced from the proof of Proposition 4.4 below (see the remark below the proof) that if θ is the inner factor of b , then such a space $\mathcal{H}(b)$ will contain a non-zero function in the class X_α if and only if K_θ will contain one, since $K_\theta \cap X_\alpha = \mathcal{H}(b) \cap X_\alpha$ in that case.

The previous example shows how the structure of the carrier of Δ can disallow smooth approximations. The size of Δ also plays a role. More precisely, Volberg constructed a very small weight W on \mathbb{T} with the property that $W > 0$ almost everywhere on \mathbb{T} , $\log W$ is not integrable on any arc of \mathbb{T} , and showed that M_z on $\mathcal{P}^2(\mu)$ with $d\mu = dA + Wdm$ contains $L^2(Wdm)$ as a direct summand. This example is contained in the editor's commentary to [11]. Meanwhile, it is worth mentioning that if $\log W$ is integrable on some arc of \mathbb{T} , then $L^2(Wdm)$ is not a direct summand of $\mathcal{P}^2(\mu)$ (see Theorem D, [11]). By re-scaling Volberg's weight W so that $W < 1/2$ almost everywhere on \mathbb{T} , we can then construct an outer function b with modulus satisfying $|b|^2 = 1 - W^2$, and then Theorem 1.1 implies the following.

Corollary 2.2. *There exists an outer function b with $|b| < 1$ almost everywhere on \mathbb{T} (thus in (7) we have $E = \mathbb{T}$, up to a set of Lebesgue measure zero) and such that $X_\alpha \cap \mathcal{H}(b)$ is not dense in $\mathcal{H}(b)$, for any $\alpha > 0$.*

The proof of Proposition 4.4 below will show that if the outer function b is constructed as explained above, then in fact we will have $X_\alpha \cap \mathcal{H}(b) = \{0\}$ for any $\alpha > 0$.

The third way in which smooth approximations can be prohibited is by existence of certain inner factors of b . To clarify this, we will need to introduce a decomposition which appeared notably in the work of Roberts in [17] on cyclic singular inner functions in the Bergman spaces. Namely, every singular measure ν on \mathbb{T} can be uniquely decomposed into mutually singular parts, one concentrated on a countable union of Beurling-Carleson sets of measure zero and one part which vanishes on such sets. More explicitly,

$$\nu = \nu_{\mathcal{C}} + \nu_{\mathcal{K}} \tag{8}$$

and there exists a set $F = \cup_k F_k$ with each F_k being Beurling-Carleson with $|F_k| = 0$ such that $\nu_{\mathcal{C}}(B) := \nu_{\mathcal{C}}(B \cap F)$ for any Borel set $B \subset \mathbb{T}$, and $\nu_{\mathcal{K}}$ assigns no mass to any Beurling-Carleson set of Lebesgue measure zero. The decomposition in (8) corresponds to a factorization $S_\nu = S_{\nu_{\mathcal{C}}} S_{\nu_{\mathcal{K}}}$. In the class of model spaces, it is the factor $S_{\nu_{\mathcal{K}}}$ that prohibits smooth approximations

(see [15]). The situation is different for general $\mathcal{H}(b)$ -spaces, and we will see below that it is the location of the mass of $\nu_{\mathcal{K}}$ that plays a decisive role.

Corollary 2.3. *Let F be the (closed) support of Δ . If S_{ν} is the inner factor of b and $\nu_{\mathcal{K}}(\mathbb{T} \setminus F) > 0$, then $X_{\alpha} \cap \mathcal{H}(b)$ is not dense in $\mathcal{H}(b)$, for any $\alpha > 0$.*

This result follows from cyclicity results of [14]. The condition $\nu_{\mathcal{K}}(\mathbb{T} \setminus F) > 0$ implies that S_{ν} contains an inner factor which is a cyclic vector for M_z on $\mathcal{P}^2(\mu)$, and thus condition (ii) of Theorem 1.1 on $[S_{\nu}]$ cannot be satisfied. Note that the set F is the *support* of Δ , which can indeed be larger than the carrier set E . If E is a Beurling-Carleson set, then $F = E$.

We now turn to positive results. We shall construct a very large class of symbols b for which $X_{\alpha} \cap \mathcal{H}(b)$ is dense in $\mathcal{H}(b)$, for any $\alpha > 0$. In particular, the class will contain all b which extend continuously to the boundary and the inner factor S_{ν} of b is such that the mass of $\nu_{\mathcal{K}}$ is placed appropriately and accordingly with the restriction given in Corollary 2.3. The class is however much larger and the recipe is as follows.

- (i) Let $\{E_k\}_k$ be a sequence of Beurling-Carleson subsets of \mathbb{T} of positive measure, $E = \cup_k E_k$ and b_0 be any outer function bounded by 1, satisfying $|b_0| = 1$ on $\mathbb{T} \setminus E$ and

$$\int_{E_k} \log(1 - |b_0|^2) dm > -\infty$$

for each set E_k ,

- (ii) $\nu_{\mathcal{K}}$ be a singular measure which vanishes on Beurling-Carleson sets of measure zero and for which we have $\nu_{\mathcal{K}}(\mathbb{T} \setminus (\cup_n E_n)) = 0$, with $\{E_k\}_k$ being the sets in (i),
- (iii) $\nu_{\mathcal{C}}$ be an arbitrary singular measure concentrated on a countable union of Beurling-Carleson sets of measure zero,
- (iv) B be an arbitrary convergent Blaschke product.

Note how the first two conditions in different ways ensure that the possible "bad" behaviour of b which appears in Corollary 2.1, Corollary 2.2 and Corollary 2.3 is avoided.

Corollary 2.4. *Let $b = BS_{\nu_{\mathcal{C}} + \nu_{\mathcal{K}}} b_0$ be constructed according to the above specifications. Then for each $\alpha > 0$, the set $X_{\alpha} \cap \mathcal{H}(b)$ is dense in $\mathcal{H}(b)$.*

This result follows from [14, Theorem 1.2], where condition (ii) of Theorem 1.1 is verified for the associated measures $\mu(b, \alpha)$. Note the similarity with the case of non-extreme b . In that case $\log(1 - |b|^2)$ is integrable on the entire circle \mathbb{T} , and then smooth approximations are certainly possible, since then even the analytic polynomials are dense in the space. If the weight $(1 - |b|^2)$ lives only on a part of the circle, then b is an extreme point and thus the set of polynomials contained in $\mathcal{H}(b)$ forms only a finite dimensional subspace. However, if the subset of \mathbb{T} on which the weight $(1 - |b|^2)$ lives satisfies our structural assumption, and is log-integrable there, then approximations by functions from smoothness classes X_α are possible (at least when b is outer), similarly to the non-extreme case.

Before ending this section and proceeding with the proof of Theorem 1.1, we would like to comment on certain related question that naturally occur in this context. Note that Corollary 2.4 shows that we can approximate with functions of class X_α , for each $\alpha > 0$. However, it turns out that one can extend Corollary 2.4 to hold for smooth functions in the class $\mathcal{A}^\infty = \bigcap_{\alpha > 0} X_\alpha$, that is, by functions analytic in \mathbb{D} with boundary values in C^∞ . This extension is non-trivial and requires a different set of tools, thus we refer the reader to [13] for a detailed treatment. Given our examples, another natural question is if the parameter $\alpha > 0$ plays a significant role, or if density of $X_\alpha \cap \mathcal{H}(b)$ in $\mathcal{H}(b)$ for small α is equivalent to density for all α , just as it is in K_θ (see [15]).

A third question that occurs in relation to Corollary 2.4 is what happens if a carrier set E of Δ is not (up to sets of Lebesgue measure zero) a countable union of Beurling-Carleson sets? In that case one can perform the following decomposition. Consider the quantity

$$\gamma = \sup_F |F|$$

where F is a Beurling-Carleson subset of E . If $\{F_n\}_n$ is a sequence of Beurling-Carleson subsets of E for which we have $\gamma = \lim_n |F_n|$, then the set $E \setminus (\cup_n F_n)$ has non-zero Lebesgue measure (since else $E \setminus N = \cup_n F_n$ for a set N of measure Lebesgue zero) and certainly satisfies the property appearing in Corollary 2.1. The example in [10], on which Corollary 2.1 is based, hints that the operator M_z on $\mathcal{P}^2(\mu)$ could in this case have a non-trivial isometric part. Indeed such a result was recently established in [16], showing that if M_z is completely non-isometric on $\mathcal{P}^2(\mu)$, then any carrier set E of Δ must necessarily be a countable union of Beurling-Carleson sets,

disregarding sets of Lebesgue measure zero.

3 X_α -SPACES AND THEIR DUALITY

In the following sections we prove Theorem 1.1, and we start by recalling some simple properties of the spaces X_α .

Recall, for $\alpha > 0$, the definition of the space X_α in (2). Naturally, we could also allow the parameter α to take on any real value, instead of requiring it be positive as in the earlier sections. We will need this broader range of parameters, and we will now describe a simple but crucial duality relation and a norm equivalence.

The spaces X_α are Hilbert spaces, but instead of seeing X_α as dual of itself with respect to the inner product, we shall instead find use of the *Cauchy duality* between X_α and $X_{-\alpha}$. The Cauchy duality is realized by

$$\lim_{r \rightarrow 1} \int_{\mathbb{T}} f_r \overline{g_r} dm, \quad f \in X_\alpha, g \in X_{-\alpha}, \quad (9)$$

where $f_r(z) = f(rz)$ and $g_r(z) = g(rz)$ are the usual dilations of f and g . Indeed, if $\{f_n\}_{n \geq 0}$ and $\{g_n\}_{n \geq 0}$ are the sequences of Taylor coefficients of f and g , then we easily see that for the above limit we have

$$\begin{aligned} \lim_{r \rightarrow 1} \sum_{n=0}^{\infty} f_n \overline{g_n} r^{2n} &= \lim_{r \rightarrow 1} \sum_{n=0}^{\infty} (n+1)^{\alpha/2} f_n \frac{\overline{g_n}}{(n+1)^{\alpha/2}} r^{2n} \\ &= \sum_{n=0}^{\infty} (n+1)^{\alpha/2} f_n \frac{\overline{g_n}}{(n+1)^{\alpha/2}} \end{aligned} \quad (10)$$

where passage to the limit $r = 1$ is justified by the fact that $\{(n+1)^{\alpha/2} f_n\}_n$ and $\{(n+1)^{-\alpha/2} g_n\}_n$ are square-summable sequences by definition of X_α and $X_{-\alpha}$. Under this pairing $X_{-\alpha}$ is isometrically isomorphic to the dual space of X_α . Clearly the Cauchy dual of $X_{-\alpha}$ is X_α , and thus the usual weak-star topology on $X_{-\alpha}$ coincides with the weak topology on $X_{-\alpha}$.

Again restricting the parameter α to be positive, a computation shows that for α fixed, we have that

$$\int_{\mathbb{D}} |z|^{2n} (1 - |z|^2)^{\alpha-1} dA(z) \simeq \frac{1}{n^\alpha}.$$

This implies immediately that the square of the norm on $X_{-\alpha}$ is equivalent to

$$\int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^{\alpha-1} dA(z).$$

4 ISOMETRIC PART OF M_z PROHIBITS SMOOTH APPROXIMATION

In this section, we shall treat one of the implications in Theorem 1.1. Namely, if the symbol b and $\alpha > 0$ is such that the corresponding $\mathcal{P}^2(\mu)$ -space contains a non-trivial subspace on which M_z acts as an isometric operator, then $X_\alpha \cap \mathcal{H}(b)$ is not dense in $\mathcal{H}(b)$. One of our main tools for this will be the following structure theorem for $\mathcal{H}(b)$ -spaces in the extreme case. The result appears in [1], and is generalized to other classes of spaces in [2]. The symbol P_+ denotes the orthogonal projection from L^2 of the circle \mathbb{T} onto the Hardy space H^2 .

Proposition 4.1. *Let b be an extreme point of the unit ball of H^∞ ,*

$$E = \{\zeta \in \mathbb{T} : |b(\zeta)| < 1\},$$

and let $\Delta = \sqrt{1 - |b|^2}$ be a function on the circle \mathbb{T} , defined in terms of boundary values of b on \mathbb{T} . For $f \in \mathcal{H}(b)$ the equation

$$P_+ \bar{b} f = -P_+ \Delta g$$

has a unique solution $g \in L^2(E)$, and the map $J : \mathcal{H}(b) \rightarrow H^2 \oplus L^2(E)$ defined by

$$Jf = (f, g),$$

is an isometry. Moreover,

$$J(\mathcal{H}(b))^\perp = \left\{ (bh, \Delta h) : h \in H^2 \right\}.$$

We will also need the following classical result on the structure of M_z -invariant subspaces of $L^2(\mathbb{T})$. For a proof, see for instance [9, Lecture II, Theorems 2 and 3].

Proposition 4.2. *Let M_z^* be the adjoint of the shift on $L^2(E)$. The M_z^* -invariant subspaces of $L^2(E)$ are of the form*

$$L^2(F) = \{f \in L^2(E) : f = 0 \text{ almost everywhere on } \mathbb{T} \setminus F\}$$

where F is a measurable subset of E , or of the form

$$U\overline{H^2} = \{U\bar{f} : f \in H^2\}$$

where U is a unimodular function.

The second type in of subspace in Proposition 4.2 can only appear if $\mathbb{T} \setminus E$ has Lebesgue measure zero, since any function $f \in U\overline{H^2}$ satisfies $f(\zeta) \neq 0$ for almost every $\zeta \in \mathbb{T}$, unless f is trivially 0.

The following observation is crucial.

Lemma 4.3. *If J is the embedding in Proposition 4.1, then*

$$Jk_b(\lambda, z) = \left(k_b(\lambda, z), \frac{-\Delta\overline{b(\lambda)}}{1 - \overline{\lambda}z} \right),$$

and if $b(\lambda) \neq 0$, then the M_z^* -invariant subspace of $L^2(E)$ generated by $\frac{-\Delta\overline{b(\lambda)}}{1 - \overline{\lambda}z}$ equals $L^2(E)$.

Proof. A simple computation involving Proposition 4.1 will reveal that the tuple $Jk_b(\lambda, z)$ has the indicated form. Note that $\frac{-\Delta\overline{b(\lambda)}}{1 - \overline{\lambda}z}$ is non-zero almost everywhere on E , and moreover, this function is not log-integrable. If a non-zero function is contained in $U\overline{H^2}$, then it is log-integrable, and subspaces of the form $L^2(F)$ contain functions vanishing on sets of positive measure unless $F = E$. The claim follows from Proposition 4.2. \square

We will use the above corollary in the proof of the following proposition, which is our first step towards proving Theorem 1.1.

Proposition 4.4. *Let b be an extreme point of the unit ball of H^∞ , $\alpha > 0$ and μ be given by (5). If $M_z : \mathcal{P}^2(\mu) \rightarrow \mathcal{P}^2(\mu)$ admits a non-trivial invariant subspace on which M_z acts as an isometry, then the set $X_\alpha \cap \mathcal{H}(b)$ is not dense in $\mathcal{H}(b)$.*

Proof. Note that if $f \in \mathcal{P}^2(w_{\alpha-1}dA)$ is non-zero, then the norm of $M_z f$ is strictly smaller than the norm of f , since $|zf(z)| < |f(z)|$ for each $z \in \mathbb{D}$. Therefore, since $\mathcal{P}^2(\mu) \subseteq \mathcal{P}^2(w_{\alpha-1}dA) \oplus L^2(\Delta^2 dm)$, the assumption implies that there exists a tuple of the form $(0, t) \in \mathcal{P}^2(\mu)$, where $t \in L^2(\Delta^2 dm)$ is non-zero. Let S be the M_z -invariant subspace of $L^2(\Delta^2 dm)$ generated by t . Then $\{0\} \oplus S$ is contained in $\mathcal{P}^2(\mu)$. The operator $h \mapsto \Delta h$ is a unitary

mapping between $L^2(\Delta^2 dm)$ and $L^2(E)$, and this mapping commutes with the shift operators on the spaces. It follows that $\Delta S = \{h\Delta : h \in S\}$ is an M_z -invariant subspace of $L^2(E)$.

For each $h \in S$ there exists a sequence of polynomials $\{p_n\}_{n \geq 1}$ such that $(p_n, p_n) \rightarrow (0, h)$ in the norm of $\mathcal{P}^2(w_{\alpha-1} dA) \oplus L^2(\Delta^2 dm)$. By multiplying the first coordinate by b and using the norm equivalence presented in Section 3, we clearly are in the following situation:

$$\|bp_n\|_{-\alpha} \rightarrow 0 \tag{11}$$

and

$$\int_E |\Delta p_n - \Delta h|^2 dm \rightarrow 0. \tag{12}$$

Assume that s is a function in $X_\alpha \cap \mathcal{H}(b)$ and consider $Js = (s, g)$, where J is the embedding of Proposition 4.1. By the description of the annihilator of $J\mathcal{H}(b)$ from same theorem, we have that

$$0 = \int_{\mathbb{T}} s \overline{bp_n} dm + \int_E g \Delta \overline{p_n} dm.$$

For the first term on the right-hand side above, we have

$$\left| \int_{\mathbb{T}} s \overline{bp_n} dm \right| \leq \|s\|_\alpha \|bp_n\|_{-\alpha} \tag{13}$$

which follows from (9), (10), (2) and Cauchy-Schwarz inequality. Thus, by (11) and (12), in the limit as $n \rightarrow \infty$ we obtain

$$0 = \int_E g \Delta \bar{h} dm.$$

Since $h \in S$ was arbitrary, we conclude that g is orthogonal to ΔS in $L^2(E)$. Since $\Delta S \neq \{0\}$, the second component g of Js is thus contained in the proper subspace $(\Delta S)^\perp$ of $L^2(E)$, whenever $s \in X_\alpha \cap \mathcal{H}(b)$. Note that $(\Delta S)^\perp$ is invariant for M_z^* . It follows now from Lemma 4.3 that tuples of the form Js cannot approximate tuples of the form $Jk_b(\lambda, z)$ whenever $b(\lambda) \neq 0$, since then the second component of $Jk_b(\lambda, z)$ does not lie in $(\Delta S)^\perp$. \square

We remark that if the invariant subspace on which M_z acts as an isometry in Proposition 4.4 is the space $L^2(\mu|\mathbb{T})$, then it follows from the proof that we have $Js = (s, 0)$ for any $s \in X_\alpha$. It follows then from Proposition 4.1 that $s \in K_\theta$, where θ is the inner factor of b . If b is outer, then $X_\alpha \cap \mathcal{H}(b) = \{0\}$.

5 NECESSITY OF THE DIVISION PROPERTY

Before going into the proof of the main result of this section, we make the following observation. By developments of Section 4, in order to prove Theorem 1.1, we may assume below that M_z is completely non-isometric on $\mathcal{P}^2(\mu)$, and thus $\mathcal{P}^2(\mu)$ is a space of analytic functions. For any $f \in H^2$, let $\{p_n\}$ be a sequence of polynomials which converges to f in H^2 and almost everywhere on \mathbb{T} . It is clear that this sequence will also be a Cauchy sequence in our $\mathcal{P}^2(\mu)$ space, and thus f will be a member of $\mathcal{P}^2(\mu)$. The part of f which lives on \mathbb{T} will be represented in $\mathcal{P}^2(\mu)$ by the usual non-tangential boundary values of f . Moreover, any bounded analytic function is a multiplier of $\mathcal{P}^2(\mu)$, and it acts in the natural way on the boundary.

The following result, together with the previous one, will complete the proof of the implication (i) \implies (ii) in Theorem 1.1.

Proposition 5.1. *Let b be an extreme point of the unit ball of H^∞ , $\alpha > 0$ and μ be given by (5). Further, let θ be the inner factor of b . Assume that $X_\alpha \cap \mathcal{H}(b)$ is dense in $\mathcal{H}(b)$. Then $M_z : \mathcal{P}^2(\mu) \rightarrow \mathcal{P}^2(\mu)$ is completely non-isometric, and moreover whenever $f \in \mathcal{N}^+$ is contained in $[\theta]$, then $f/\theta \in \mathcal{N}^+$.*

Proof. By the above discussion, M_z is completely non-isometric, $\mathcal{P}^2(\mu)$ is a space of analytic functions, and it remains to show that the division property in $[\theta]$ holds. Assume therefore that $\lim_{n \rightarrow \infty} \|\theta p_n - f\|_{\mu,2} = 0$, where $f = u/v \in \mathcal{N}^+$, u, v are bounded analytic functions with v being outer, and $\{p_n\}_{n \geq 1}$ is a sequence of bounded analytic functions. By multiplying the convergent sequence by v we can assume that f is bounded. Let b_0 be the outer factor of b . By further multiplication by b_0 we deduce from the above convergence that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{D}} |bp_n - b_0 f|^2 w_{\alpha-1} dA = 0,$$

and we also have that

$$\lim_{n \rightarrow \infty} \int_E |\Delta \theta p_n - \Delta f|^2 dm = \lim_{n \rightarrow \infty} \int_E |\Delta p_n - \Delta \bar{\theta} f|^2 dm = 0.$$

Now, take any $s \in X_\alpha \cap \mathcal{H}(b)$. If $Js = (s, g)$, then the above two limits being zero can be used to show, just as in the proof of Proposition 4.4, that

$$\int_{\mathbb{T}} s \bar{b}_0 \bar{f} dm + \int_E g \Delta \bar{\theta} \bar{f} dm = \lim_{n \rightarrow \infty} \left(\int_{\mathbb{T}} s \bar{b} \bar{p}_n dm + \int_E g \Delta \bar{p}_n dm \right) = 0.$$

The last equality follows by the orthogonality relation of Proposition 4.1. By the density of the elements of the form J_s in $J\mathcal{H}(b)$, this implies that $(b_0f, \Delta\bar{\theta}f) \perp J\mathcal{H}(b)$, and so $(b_0f, \Delta\bar{\theta}f) = (bh, \Delta h)$ for some $h \in H^2$, again by Proposition 4.1. Thus $f/\theta = h \in \mathcal{N}^+$, and the proof is complete. \square

6 SUFFICIENCY OF THE CONDITIONS ON M_z FOR APPROXIMATION

We prove now the implication (ii) \implies (i) of Theorem 1.1.

Proposition 6.1. *Let b be an extreme point of the unit ball of H^∞ , $\alpha > 0$ and μ be given by (5). Further, let θ be the inner factor of b . Assume that $M_z : \mathcal{P}^2(\mu) \rightarrow \mathcal{P}^2(\mu)$ admits no non-trivial invariant subspace on which M_z acts as an isometric operator, and moreover whenever $f \in \mathcal{N}^+$ is in $[\theta]$, then $f/\theta \in \mathcal{N}^+$. Then $X_\alpha \cap \mathcal{H}(b)$ is dense in $\mathcal{H}(b)$.*

Proof. Let us assume that $f \in \mathcal{H}(b)$ is orthogonal to $X_\alpha \cap \mathcal{H}(b)$. We will show that the assumptions in the statement of the proposition imply that $f = 0$. Because the mapping J in Proposition 4.1 is an isometry, it follows that Jf is orthogonal to $J(X_\alpha \cap \mathcal{H}(b))$. Note that $J(X_\alpha \cap \mathcal{H}(b))$ is a subspace of $X_\alpha \oplus L^2(E)$.

We claim that, under the Cauchy duality between X_α and $X_{-\alpha}$, we have

$$J(X_\alpha \cap \mathcal{H}(b)) = \bigcap_{h \in H^2} \ker l_h, \tag{14}$$

where l_h is the functional on $X_\alpha \oplus L^2(E)$ given by Cauchy pairing with $(bh, \Delta h) \in X_{-\alpha} \oplus L^2(E)$, $h \in H^2$. Indeed, any pair $(c, d) \in X_\alpha \oplus L^2(E)$ annihilated by $\{l_h\}_{h \in H^2}$ lies in $J(\mathcal{H}(b))$ by Proposition 4.1. Conversely, again by same formula, any function in $J(X_\alpha \cap \mathcal{H}(b))$ will be annihilated by $\{l_h\}_{h \in H^2}$.

Equip $X_{-\alpha} \oplus L^2(E)$ with the weak*-topology as the dual space of $X_\alpha \oplus L^2(E)$. The fact that Jf annihilates $J(X_\alpha \cap \mathcal{H}(b))$ and (14) holds implies that Jf is contained in the weak*-closure of the linear manifold $\{l_h\}_{H^2} \subseteq X_{-\alpha} \oplus L^2(E)$. Since the dual of $X_{-\alpha}$ is X_α and vice versa, it follows that the weak*-topology coincides with the weak-topology on $X_{-\alpha} \oplus L^2(E)$. Thus Jf is contained in the weak-closure of $\{l_h\}_{h \in H^2}$. Since $\{l_h\}_{h \in H^2}$ is a convex set, its weak-closure and norm-closure coincide. Thus there exists a sequence $\{h_k\}_{k \geq 1}$ with $h_k \in H^2$ such that

$$(bh_k, \Delta h_k) \rightarrow Jf := (f, g) \tag{15}$$

in the norm of $X_{-\alpha} \oplus L^2(E)$. It follows from (15) by multiplication of the second coordinate by b , and from norm equality presented in Section 3, that $\{bh_k\}_k$ is a Cauchy sequence in $\mathcal{P}^2(\mu)$, and thus converges to a function in $\mathcal{P}^2(\mu)$, which of course must be f . We now use the assumption that M_z is completely non-isometric: since f is a function in H^2 , the discussion in the beginning of Section 5 implies that $bg = \Delta f$ and thus that $g = \Delta f/b$ on E . Moreover, $Jf = (f, \Delta f/b)$, and by our assumption $f/\theta \in \mathcal{N}^+$, thus actually $f/\theta \in H^2$ by the Smirnov maximum principle. Since we are assuming that $f \in \mathcal{H}(b)$, by Proposition 4.1 we get that

$$0 = P_+(\bar{b}f + \Delta g) = P_+(\bar{b}f + \Delta^2 f/b) = P_+(|b|^2 f/b + \Delta^2 f/b) = P_+(f/b). \quad (16)$$

In the above computation we showed that $\bar{b}f + \Delta g = f/b$ on \mathbb{T} , and thus f/b has square-integrable boundary values. Since $f/\theta \in H^2$, it follows from the Smirnov maximum principle that $f/b \in H^2$. Thus f/b is an analytic function which projects to 0 under P_+ , which implies that $f/b = 0$, and consequently $f = 0$. \square

Note that, together with developments of Section 4, the proof of Theorem 1.1 is now complete.

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