Cyclic inner functions in growth classes and applications to approximation problems

Bartosz Malman

Abstract

It is well-known that for any inner function θ defined in the unit disk $\mathbb{D}$ the following two conditons: (i) there exists a sequence of polynomials $\{p_n\}_n$ such that $\lim_{n\to\infty} \theta(z)p_n(z) = 1$ for all $z \in \mathbb{D}$, and (ii) $\sup_n \|\theta p_n\|_\infty < \infty$, are incompatible, i.e., cannot be satisfied simultaneously. However, it is also known that if we relax the second condition to allow for arbitrarily slow growth of the sequence $\{\theta(z)p_n(z)\}_n$ as $|z| \to 1$, then condition (i) can be met for some singular inner function. We discuss certain consequences of this fact which are related to the rate of decay of Taylor coefficients and moduli of continuity of functions in model spaces K_θ. In particular, we establish a variant of a result of Khavinson and Dyakonov on non-existence of functions with certain smoothness properties in K_θ, and we show that the classical Aleksandrov theorem on density of continuous functions in K_θ, is essentially optimal. We consider also the same questions in the context of de Branges-Rovnyak spaces $\mathcal{H}(b)$, and show that the corresponding approximation result also is optimal.

1 Background and the main results

1.1 Cyclic singular inner functions

Let X be a topological space consisting of functions which are analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and which satisfies some customary desirable properties, such as that the evaluation $f \mapsto f(\lambda)$ is a continuous functional on X for each $\lambda \in \mathbb{D}$, and that the function $z \mapsto zf(z)$ is contained in the space X whenever $f \in X$. A function $g \in X$ is said to be *cyclic* if there exists a sequence of analytic polynomials $\{p_n\}_n$ for which the polynomial multiples $\{gp_n\}_n$ converge to the constant function 1 in the topology of the space.

The well-known Hardy classes H^p are among the very few examples of analytic function spaces in which the cyclicity phenomenon is completely understood. The cyclic functions g are of the form

$$g(z) = \exp\Big(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log(|g(\zeta)|)\, dm(\zeta)\Big), \quad z \in \mathbb{D}, \tag{1}$$

where dm is the (normalized) Lebesgue measure of the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Functions as in (1) are called *outer functions*. The *inner functions* are of the form

$$\begin{aligned}\theta(z) &= B(z)S_\nu(z) \\ &= \prod_n \frac{\overline{\alpha_n}}{|\alpha_n|} \frac{\alpha_n - z}{1 - \overline{\alpha_n} z} \cdot \exp\Big(- \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\nu(\zeta)\Big), \quad z \in \mathbb{D}, \end{aligned} \tag{2}$$

where ν is a positive finite singular Borel measure on $\mathbb{T}$ and $\{\alpha_n\}_n$ is a Blaschke sequence. It is clear that if the Blaschke product B on the left is non-trivial, then θ vanishes at points in $\mathbb{D}$ and therefore cannot be cyclic in any reasonable space of analytic functions X. The right factor S_ν is a *singular inner function*, and it is well-known that if a function $g \in H^p$ has a singular inner function as a factor, then g is not cyclic in H^p. As a consequence, if $\{p_n\}_n$ is a sequence of polynomials for which we have

$$\lim_{n\to\infty} \theta(z)p_n(z) = 1, \quad z \in \mathbb{D},$$

then necessarily the Hardy class norms of the sequence must explode:

$$\lim_{n\to\infty} \|\theta p_n\|^p_{H^p} := \lim_{n\to\infty} \int_{\mathbb{T}} |\theta p_n|^p dm = \infty$$

for finite $p \geq 1$, or in case $p = \infty$,

$$\lim_{n\to\infty} \|\theta p_n\|_\infty := \lim_{n\to\infty} \sup_{z\in\mathbb{D}} |\theta(z)p_n(z)| = \infty.$$

When other norms are considered, cyclic singular inner functions might exist, and here the Bergman spaces $L^p_a(\mathbb{D})$ provide a famous set of examples. The Bergman norms are of the form

$$\|g\|^p_{L^p(\mathbb{D})} := \int_{\mathbb{D}} |g(z)|^p dA(z),$$

where dA is the normalized area measure of $\mathbb{D}$. After a sequence of partial results by multiple authors, Boris Korenblum in [12] and James Roberts in [15] independently characterized the cyclic singular inner functions in the Bergman spaces in terms of the vanishing on certain subsets of $\mathbb{T}$ of the corresponding singular measure ν appearing in (2). A construction of a singular inner function which is cyclic in the classical Bloch space appears in [3].

Recently, Thomas Ransford in [14] noted that singular inner functions exist which decay arbitrarily slowly near the boundary of the disk. As we shall see below, this fact has as a direct consequence the existence of an abundance of spaces of analytic function which admit cyclic singular inner functions. Here is the precise statement of the main result of [14].

Theorem 1.1. *Let $w : [0, 1) \to (0, 1)$ be any function satisfying $\lim_{r\to1^-} w(r) = 0$. Then there exists a singular inner function θ for which we have*

$$\min_{|z|<r} |\theta(z)| \geq w(r), \quad r \in (0, 1). \tag{3}$$

It has been remarked to the present author that, in fact, this theorem appears already in the literature. For instance, Harold Shapiro similarly mentions in [17] that a singular inner function always satisfies an estimate of the form

$$|S_\nu(z)| \geq \exp\Big(- C\frac{\omega(1-|z|)}{1-|z|}\Big), \tag{4}$$

where C is some positive constant, and $\omega = \omega_\nu$ is the modulus of continuity of the measure ν:

$$\omega_\nu(h) = \sup_{|I|=h} \nu(I). \tag{5}$$

The supremum above is taken over arcs I of the circle $\mathbb{T}$ which are of length h. In [18] Shapiro proves that a singular measure ν exists with a modulus of continuity ω_ν for which $\omega_\nu(h)/h$ grows to infinity arbitrarily slowly as h decreases to zero, hence proving Theorem 1.1 as a consequence of the estimate (4). In fact, such singular measures have been known to exist at least since the work of Hartman and Kershner in [10]. The proof of Ransford in [14] also involves establishing the existence of such a measure.

The following result is the above mentioned consequence of Theorem 1.1 on existence of cyclic singular inner function. The result is surely well-known, and has an elementary proof which we include for convenience.

Corollary 1.2. *Let $w : [0,1) \to (0,1)$ be any decreasing function satisfying $\lim_{t\to 1^-} w(t) = 0$. There exists a singular inner function $\theta = S_\nu$ and a sequence of analytic polynomials $\{p_n\}_n$ such that*

(i) $\lim_{n\to\infty} \theta(z)p_n(z) = 1, \quad z \in \mathbb{D}$,

(ii) $\sup_{z\in\mathbb{D}} |\theta(z)p_n(z)|w(|z|) \leq 2$.

Proof. Apply Theorem 1.1 to the function w to produce a singular inner function θ satisfying (3). For integers $n \geq 2$ we set $r_n := 1 - 1/n$ and $Q_n(z) := 1/\theta(r_n z)$. Then Q_n is holomorphic in a neighbourhood of the closed disk $\overline{\mathbb{D}}$, and because we are assuming that w is decreasing, we have the estimate

$$\sup_{z\in\mathbb{D}} |Q_n(z)|w(|z|) \leq \sup_{z\in\mathbb{D}} \frac{w(|z|)}{w(r_n|z|)} \leq 1.$$

We can approximate Q_n by an analytic polynomial p_n so that

$$\sup_{z\in\mathbb{D}} |Q_n(z) - p_n(z)| \leq 1/n.$$

Then

$$\sup_{z\in\mathbb{D}} |\theta(z)p_n(z)|w(|z|) \leq \sup_{z\in\mathbb{D}} \Big(|\theta(z)Q_n(z)| + 1/n\Big)w(|z|) \leq 2$$

It is clear from the construction that $\theta(z)p_n(z) \to 1$ as $n \to \infty$, for any $z \in \mathbb{D}$. $\square$

Corollary 1.2 says that there exists cyclic singular inner functions in essentially any space of analytic functions defined in terms of a growth condition, or in any space in which such a *growth space* is continuously embedded.

The purpose of this note is to apply Theorem 1.1, or more precisely its simple consequence stated in Corollary 1.2, to the questions of existence of functions with certain smoothess properties in model spaces K_θ. We will establish sharpness of certain existing approximation results in these spaces. Moreover, we take the opportunity to discuss similar questions in the broader class of de Branges-Rovnyak spaces $\mathcal{H}(b)$. Our results are proved by rather well-known methods, but their statements seem to be missing in the existing literature, and we wish to fill in this gap.

In the proofs of the main results, which will be stated shortly, we will concern ourselves with the following *weak* type of cyclicity of singular inner functions. Let Y be some linear space of analytic functions which is contained in H^1. We want to investigate if there exists a singular inner function θ and a sequence of polynomials $\{p_n\}_n$ such that

$$f(0) = \int_{\mathbb{T}} f\, dm = \lim_{n\to\infty} \int f\overline{\theta p_n}\, dm \tag{6}$$

holds for all $f \in Y$. The above situation means that the sequence $\{\theta p_n\}_n$ converges to the constant 1, weakly over the space Y. Now, clearly if Y is too large of a space (say, $Y = H^2$), then (6) can never hold for all $f \in Y$. However, if Y is sufficiently small, then the situation in (6) might occur. For instance, in the extreme case when Y is a set of analytic polynomials, then any singular inner function θ and any sequence of polynomials $\{p_n\}_n$ which satisfies $\lim_{n\to\infty} p_n(z) = 1/\theta(z)$ for $z \in \mathbb{D}$, is sufficient to make (6) hold. Philosophically speaking, it is the uniform smoothness of the functions in the class Y that allows the existence of singular inner functions θ for which the above situation occurs. Under insignificant assumptions on Y, a straightforward argument shows that if (6) occurs, then the intersection between Y and K_θ is trivial, while Corollary 1.2 provides us with a huge class of spaces Y for which (6) can be achieved.

1.2 Main results

Recall that the space K_θ is constructed from an inner function θ by taking the orthogonal complement of the subspace

$$\theta H^2 := \{\theta h : h \in H^2\}$$

in the Hardy space H^2:

$$K_\theta = H^2 \ominus \theta H^2.$$

For background on the spaces K_θ one can consult the books [5] and [9]. In our first result, we will show that the famous approximation theorem of Aleksandrov from [1] on density in K_θ of functions which extend continuously to the boundary, is in fact essentially sharp, as it cannot be extended to any class of functions satisfying an estimate on their modulus of continuity. By a *modulus of continuity* ω we mean here a function $\omega : [0, \infty) \to [0, \infty)$

which is continuous, increasing, satisfies $\omega(0) = 0$, and for which $\omega(t)/t$ is a decreasing function with

$$\lim_{t \to 0^+} \omega(t)/t = \infty.$$

For such a function ω we define Λ_a^ω to be the space of functions f which are analytic in $\mathbb{D}$, extend continuously to $\overline{\mathbb{D}}$, and satisfy

$$\sup_{z,w \in \overline{\mathbb{D}}, z \neq w} \frac{|f(z) - f(w)|}{\omega(|z-w|)} < \infty. \tag{7}$$

Then Λ_a^ω is the space of analytic functions on $\mathbb{D}$ which have a modulus of continuity dominated by ω. We make Λ_a^ω into a normed space by introducing the quantity

$$\|f\|_\omega := \|f\|_\infty + \sup_{z,w \in \overline{\mathbb{D}}, z \neq w} \frac{|f(z) - f(w)|}{\omega(|z-w|)}.$$

By a theorem of Tamrazov from [19], we could have replaced the supremum over $\overline{\mathbb{D}}$ by a supremum over $\mathbb{T}$, and obtain the same space of functions (we remark that a nice proof of this result is contained in [4, Appendix A]). The following is an optimality statement regarding Aleksandrov's density theorem.

Theorem 1.3. *Let ω be a modulus of continuity. There exists a singular inner function θ such that*

$$\Lambda_a^\omega \cap K_\theta = \{0\}.$$

This statement will be proved in Section 3. In fact, we will see that Theorem 1.3 is a consequence of a variant, and in some directions a strengthening, of a theorem of Dyakonov and Khavinson from [6]. For a sequence of positive numbers $\boldsymbol{\lambda} = \{\lambda_n\}_{n=0}^\infty$ we define the class

$$H^2_{\boldsymbol{\lambda}} = \Big\{ f = \sum_{n=0}^\infty f_n z^n \in Hol(\mathbb{D}) : \sum_{n=0}^\infty \lambda_n |f_n|^2 < \infty \Big\}. \tag{8}$$

The next theorem, proved in Section 2, reads as follows.

Theorem 1.4. *Let $\boldsymbol{\lambda} = \{\lambda_n\}_{n=0}^\infty$ be any increasing sequence of positive numbers with $\lim_{n\to\infty} \lambda_n = \infty$. Then there exists a singular inner function θ such that*

$$H^2_{\boldsymbol{\lambda}} \cap K_\theta = \{0\}.$$

The result can be compared to the mentioned result of Dyakonov and Khavinson in [6], from which the above result can be deduced in the special case $\boldsymbol{\lambda} = \{(k+1)^\alpha\}_{k=0}^\infty$ with any $\alpha > 0$.

The theory of de Branges-Rovnyak spaces $\mathcal{H}(b)$ is a well-known generalization of the theory of model spaces K_θ. The symbol of the space b is now any analytic self-map of the unit disk, and we have $\mathcal{H}(b) = K_b$ whenever b is inner. For background on $\mathcal{H}(b)$ spaces one can consult [16], or [7] and [8]. A consequence of the author's work in collaboration with Alexandru Aleman in [2] is that the above mentioned density theorem of Aleksandrov generalizes to the broader class of $\mathcal{H}(b)$ spaces: any such space admits a dense subset of functions which extend continuously to the boundary. Since Theorem 1.3 proves optimality of Aleksandrov's theorem for inner functions θ, one could ask if at least for outer symbols b any improvement of the density result in $\mathcal{H}(b)$ from [2] can be obtained. In Section 4 we remark that this is not the case, and the result in [2] is also essentially optimal, even for outer symbols b.

Theorem 1.5. *Let $\boldsymbol{\lambda} = \{\lambda_n\}_{n=0}^\infty$ be any increasing sequence of positive numbers with $\lim_{n\to\infty} \lambda_n = \infty$. There exists an outer function $b : \mathbb{D} \to \mathbb{D}$ such that*

$$H^2_{\boldsymbol{\lambda}} \cap \mathcal{H}(b) = \{0\}.$$

Theorem 1.6. *Let ω be a modulus of continuity. There exists an outer function $b : \mathbb{D} \to \mathbb{D}$ such that*

$$\Lambda^\omega_a \cap \mathcal{H}(b) = \{0\}.$$

We will show that the above results are essentially equivalent to a theorem of Khrushchev from [11].

In the last Section 5 we list a few questions we have not found an answer for, and some ideas for further research.

2 Proof of Theorem 1.4

In the proof of the theorem we will need to use the following crude construction of an integrable weight with large moments.

Lemma 2.1. *Let $\{\lambda_n\}_{n=0}^\infty$ be a decreasing sequence of positive numbers with $\lim_{n\to\infty} \lambda_n = 0$. There exists a non-negative function $\Lambda \in L^1([0,1])$ which satisfies*

$$\lambda_n \leq \int_0^1 x^{2n+1} \Lambda(x)\, dx, \quad n \geq 0.$$

Proof. Recall that the sequence $(1-1/n)^n = \exp(n \log(1-1/n))$ is decreasing and satisfies

$$\lim_{n\to\infty} (1-1/n)^n = e^{-1}.$$

It follows that

$$\inf_{x \in (1-1/n, 1)} x^{2n+1} \geq \alpha$$

for some constant $\alpha > 0$ which is independent of n. For $n \geq 1$, we define the intervals $I_n = \big(1-1/n, 1-1/(n+1)\big)$. Our function Λ will be chosen to be of the form

$$\Lambda(x) = \sum_{n=0}^\infty 1_{I_n} c_n,$$

where 1_{I_n} is the indicator function of the interval I_n and the c_n are positive constants to be chosen shortly. Note that

$$\int_0^1 x^{2N+1}\Lambda(x)\,dx \geq \int_{1-1/N}^1 x^{2N+1}\Lambda(x)\,dx \geq \alpha \sum_{n=N}^{\infty} |I_n| c_n. \tag{9}$$

We choose

$$c_n = \alpha^{-1}|I_n|^{-1}(\lambda_n - \lambda_{n+1}).$$

This choice of coefficients c_n makes Λ integrable over $[0,1]$:

$$\int_0^1 \Lambda(x)dx = \sum_{n=1}^{\infty} |I_n| c_n = \alpha^{-1} \sum_{n=1}^{\infty} \lambda_n - \lambda_{n+1}$$

$$= \lim_{M\to\infty} \alpha^{-1} \sum_{n=1}^{M} \lambda_n - \lambda_{n+1} = \lim_{M\to\infty} \alpha^{-1}(\lambda_1 - \lambda_{M+1})$$

$$= \alpha^{-1}\lambda_1$$

In the last step we used the assumption that the sequence $\{\lambda_n\}_n$ converges to zero. Moreover, by (9) and the choice of c_n we can estimate

$$\int_0^1 x^{2N+1}\Lambda(x)\,dx \geq \alpha \sum_{n=N}^{\infty} |I_n| c_n$$

$$= \lim_{M\to\infty} \alpha \sum_{n=N}^{M} |I_n| c_n = \lim_{M\to\infty} \sum_{n=N}^{M} \lambda_n - \lambda_{n+1}$$

$$= \lim_{M\to\infty} \lambda_N - \lambda_{M+1} = \lambda_N.$$

The proof is complete. □

The significance of the above lemma is the estimate

$$\sum_{k=0}^{\infty} \lambda_k |f_k|^2 \leq c \int_{\mathbb{D}} |f(z)|^2 \Lambda(|z|) dA(z) \tag{10}$$

for some numerical constant $c > 0$ and any function f which is holomorphic in a neighbourhood of the closed disk $\overline{\mathbb{D}}$. The estimate can be verified by direct computation of the integral on the right-hand side, using polar coordinates.

We will also use the following well-known construction.

Lemma 2.2. *For any function $g \in L^1([0,1])$ there exists a positive and increasing function $w : [0,1) \to \mathbb{R}$ which satisfies*

$$\lim_{t\to 1^-} w(t) = \infty$$

and

$$wg \in L^1([0,1]).$$

Proof. The integrability condition on g implies that

$$\lim_{t\to 1^-}\int_t^1 |g(x)|dx = 0.$$

Thus there exists a sequence of intervals $\{I_n\}_{n=1}^\infty$ which have 1 as the right end-point and length shrinking to zero, which satisfy $I_{n+1} \subset I_n$ for all $n \geq 1$, and

$$\int_{I_n} |g(x)|dx \leq 4^{-n}.$$

If we set

$$w(t) = 1_{[0,1)\setminus I_1} + \sum_{n=1}^\infty 2^n 1_{I_n\setminus I_{n+1}},$$

where $1_{I_n\setminus I_{n+1}}$ is the indicator function of the set difference $I_n \setminus I_{n+1}$, then w is increasing, satisfies $\lim_{t\to 1^-} w(t) = \infty$, and

$$\int_{I_n\setminus I_{n+1}} w(x)|g(x)|dx \leq 2^{-n}$$

for all $n \geq 1$. Consequently

$$\int_0^1 w(x)|g(x)|dx \leq \int_0^1 |g(x)|dx + \sum_n \int_{I_n\setminus I_{n+1}} w(x)|g(x)|dx < \infty.$$

□

Proof of Theorem 1.4. Let Λ be the function in Lemma 2.1 which corresponds to the sequence $\{1/\lambda_n\}_{n=0}^\infty$. That is, Λ satisfies

$$\frac{1}{\lambda_n} \leq \int_0^1 x^{2n+1}\Lambda(x)\,dx, \quad n \geq 0,$$

and $\Lambda \in L^1[0,1]$. Now, let w be a positive decreasing function which satisfies $w(x) < 1/2$, $\lim_{x\to 1^-} w(x) = 0$ and

$$\int_0^1 \frac{\Lambda(x)}{w^2(x)}\,dx < \infty.$$

Existence of such a function follows readily from Lemma 2.2. Apply Corollary 1.2 to w and obtain a corresponding inner function θ and a sequence of polynomials $\{p_n\}_n$ for which the conclusions (*i*) and (*ii*) of Corollary 1.2 hold. We will show that for this θ we have $K_\theta \cap H^2_{\boldsymbol{\lambda}} = \{0\}$.

Indeed, assume $f \in K_\theta \cap H^2_{\boldsymbol{\lambda}} = \{0\}$, but that in fact f is non-zero. Since both K_θ and $H^2_{\boldsymbol{\lambda}}$ are invariant for the backward shift operator, we may without loss of generality assume that $f(0) \neq 0$. Fix an integer n and let

$$g(z) = \theta(z)p_n(z) - 1, z \in \mathbb{D}. \tag{11}$$

Let $\{f_k\}_k, \{g_k\}_k$ be the sequences of Taylor coefficients of f and g, respectively. Since $f \in K_\theta$, we have

$$\begin{aligned} |f(0)| &= \Big|\int_{\mathbb{T}} f\, dm\Big| = \Big|\int_{\mathbb{T}} f\overline{\theta p_n - 1}\, dm\Big| = \lim_{r\to 1^-} \Big|\sum_{k=0}^\infty r^{2k} f_k \overline{g_k}\Big| \\ &\leq \limsup_{r\to 1^-} \Big(\sum_{k=0}^\infty \lambda_k r^{2k}|f_k|^2\Big)^{1/2}\Big(\sum_{k=0}^\infty \frac{1}{\lambda_k}|r^k g_k|^2\Big)^{1/2} \end{aligned}$$

Using inequality (10) on the term on the right-hand side in the last expression (with λ_n replaced by $1/\lambda_n$), we obtain

$$\begin{aligned} |f(0)| &\leq C \limsup_{r\to 1^-} \Big(\sum_{k=0}^\infty \lambda_k |f_k|^2\Big)^{1/2}\Big(\int_{\mathbb{D}} |g(rz)|^2 \Lambda(|z|) dA(z)\Big)^{1/2} \\ &= C\Big(\sum_{k=0}^\infty \lambda_k |f_k|^2\Big)^{1/2}\Big(\int_{\mathbb{D}} |g(z)|^2 \Lambda(|z|) dA(z)\Big)^{1/2}. \end{aligned}$$

By assertion in part *(ii)* of Corollary 1.2, the function $|g(z)|^2\Lambda(|z|)$ is dominated pointwise in $\mathbb{D}$ by the integrable function

$$\frac{4\Lambda(|z|)}{w^2(|z|)}, \quad z \in \mathbb{D}$$

independently of which polynomial p_n is used to defined g in (11). But if we let $n \to \infty$ in (11), then $|g(z)|^2\Lambda(|z|) \to 0$, and so we infer from the computation above and the dominated convergence theorem that $f(0) = 0$, which is a contradiction. The conclusion is that $K_\theta \cap H^2_{\boldsymbol{\lambda}} = \{0\}$, and the proof of the theorem is complete. □

3 Proof of Theorem 1.3

Theorem 1.3 will follow immediately from Theorem 1.4 together with the following embedding result for the spaces Λ^ω_a.

Lemma 3.1. *Let ω be a modulus of continuity. There exists an increasing sequence of positive numbers $\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^\infty$ satisfying $\lim_{n\to\infty} \alpha_n = \infty$ and such that for any $f \in \Lambda^\omega_a$ we have the estimate*

$$\sum_{n=0} \alpha_n |f_n|^2 \leq C\|f\|^2_\omega \tag{12}$$

where $C > 0$ is a numerical constant and $\{f_n\}_n$ is the sequence of Taylor coefficients of f.

Proof. For each $r \in (0,1)$ we have the estimate

$$\sum_{n=0} (1 - r^{2n})|f_n|^2 = \int_{\mathbb{T}} |f(\zeta) - f(r\zeta)|^2 dm(\zeta) \leq \omega(1-r)^2 \|f\|^2_\omega. \tag{13}$$

Since $\lim_{t\to 0}\omega(t) = 0$, for each positive integer N there exists a number $r_N \in (0,1)$ such that $\omega(1-r_N) \le \frac{1}{2^N}$. Since $\lim_{n\to\infty} r_N^{2n} = 0$, there exists an integer $K(N)$ such that $r_N^{2n} < 1/2$ for $n \ge K(N)$. Then

$$\sum_{n=K(N)}^{\infty} \frac{|f_n|^2}{2} \le \sum_{n=K(N)}^{\infty} (1-r_N^{2n})|f_n|^2 \le \frac{1}{4^N}\|f\|_\omega^2.$$

Consequently

$$\sum_{n=K(N)} 2^N |f_n|^2 \le \frac{1}{2^{N-1}}\|f\|_\omega^2. \tag{14}$$

We can clearly choose the sequence of integers $K(N)$ to be increasing with N. If we define the sequence $\boldsymbol{\alpha}$ by the equation $\alpha_n = 1$ for $n < K(1)$, and $\alpha_n = 2^N$ for $K(N) \le n < K(N+1)$, then (12) follows readily from (14) by summing over all $N \ge 1$. $\square$

Proof of Theorem 1.3. Lemma 3.1 implies that Λ_a^ω is contained in some space of the form $H^2_{\boldsymbol{\alpha}}$ as defined in (8). If θ is a singular inner function given by Theorem 1.4 such that $H^2_{\boldsymbol{\alpha}} \cap K_\theta = \{0\}$, then obviously we also have that $\Lambda_a^\omega \cap K_\theta = \{0\}$, and so the claim follows. $\square$

4 Proofs of Theorem 1.5 and Theorem 1.6

Here we prove the optimality of the continuous approximation theorem for the larger class of $\mathcal{H}(b)$-spaces. As remarked in the introduction, this is essentially equivalent to a theorem of Khrushchev from [11].

Proof of Theorem 1.5. By a result of Khrushchev noted in [11, Theorem 2.4], there exists a closed subset E of the circle $\mathbb{T}$ with the property that for no non-zero integrable function h supported on E is the Cauchy integral

$$C_h(z) = \int_{\mathbb{T}} \frac{h(\zeta)}{1 - z\overline{\zeta}} dm(\zeta)$$

a member of the space $H^2_{\boldsymbol{\lambda}}$. It suffices thus to construct an $\mathcal{H}(b)$ space for which every function can be expressed as such a Cauchy integral. The simplest choice for the space symbol b is the outer function with modulus 1 on $\mathbb{T} \setminus E$ and $1/2$ on E. Then b is invertible in the algebra H^∞, and a consequence of the general theory (see [8, Theorem 20.1 and Theorem 28.1]) is that every function in the space $\mathcal{H}(b)$ is a Cauchy integral of a function h which is square-integrable on $\mathbb{T}$ and supported only on E. Thus $H^2_{\boldsymbol{\lambda}} \cap \mathcal{H}(b) = \{0\}$, by Khrushchev's theorem. $\square$

Finally, Theorem 1.6 follows from Theorem 1.5 in the same way as Theorem 1.3 follows from Theorem 1.4.

5 Some ending questions and remarks

Since Theorem 1.1 seems to be such a powerful tool in establishing results of the kind mentioned here, we are wondering whether it can be further applied. In particular, the following questions come to mind.

1. Are our methods strong enough to prove that there exists model spaces K_θ which admit no non-zero functions in the Wiener algebra of absolutely convergent Fourier series? The result is known, and has been noted in [13]. However, it was proved as a consequence of a complicated construction of a cyclic singular inner function in the Bloch space. Is it so that the construction in Corollary 1.2 is sufficient to prove the non-density result for the Wiener algebra in the fashion presented here?

2. For $p > 2$, the Banach spaces ℓ^p_a consisting of functions $f \in Hol(\mathbb{D})$ with Taylor series $\{f_n\}_{n=0}^{\infty}$ satisfying

$$\|f\|^p_{\ell^p_a} := \sum_{n=0}^{\infty} |f_n|^p < \infty$$

are of course larger than the space $H^2 = \ell^2_a$. Do there exist cyclic singular inner functions in these spaces?

Acknowledgements

The author would like to thank Christopher Felder for his reading of the manuscript and for his helpful suggestions for improvements. He would also like to thank Adem Limani for very useful discussions and important comments.

References

[1] A. B. Aleksandrov. Invariant subspaces of shift operators. An axiomatic approach. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 113:7–26, 264, 1981.

[2] A. Aleman and B. Malman. Density of disk algebra functions in de Branges–Rovnyak spaces. *C. R. Math. Acad. Sci. Paris*, 355(8):871–875, 2017.

[3] J.M. Anderson, J.L. Fernandez, and A. L. Shields. Inner functions and cyclic vectors in the Bloch space. *Transactions of the American Mathematical Society*, 323(1):429–448, 1991.

[4] B. Bouya. Closed ideals in analytic weighted Lipschitz algebras. *Advances in Mathematics*, 219(5):1446–1468, 2008.

[5] J. Cima, A. Matheson, and W. Ross. *The Cauchy transform*, volume 125 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2006.

[6] K. Dyakonov and D. Khavinson. Smooth functions in star-invariant subspaces. *Contemporary Mathematics*, 393:59, 2006.

[7] E. Fricain and J. Mashreghi. *The theory of $\mathcal{H}(b)$ spaces. Vol. 1*, volume 20 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2016.

[8] E. Fricain and J. Mashreghi. *The theory of $\mathcal{H}(b)$ spaces. Vol. 2*, volume 21 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2016.

[9] S. R Garcia, J. Mashreghi, and W. T Ross. Introduction to model spaces and their operators. 2016.

[10] P. Hartman and R. Kershner. The structure of monotone functions. *American Journal of Mathematics*, 59(4):809–822, 1937.

[11] S. V. Khrushchev. The problem of simultaneous approximation and of removal of the singularities of Cauchy type integrals. *Trudy Matematicheskogo Instituta imeni VA Steklova*, 130:124–195, 1978.

[12] B. Korenblum. An extension of the Nevanlinna theory. *Acta Mathematica*, 135:187–219, 1975.

[13] A. Limani and B. Malman. An abstract approach to approximation in spaces of pseudocontinuable functions. *Proceedings of the American Mathematical Society*, 150(06):2509–2519, 2022.

[14] T. Ransford. On the decay of singular inner functions. *Canadian Mathematical Bulletin*, 64(4):902–905, 2021.

[15] J. Roberts. Cyclic inner functions in the Bergman spaces and weak outer functions in H^p, $0 < p < 1$. *Illinois Journal of Mathematics*, 29(1):25–38, 1985.

[16] D. Sarason. *Sub-Hardy Hilbert spaces in the unit disk*, volume 10 of *University of Arkansas Lecture Notes in the Mathematical Sciences*. John Wiley & Sons, Inc., New York, 1994.

[17] H. S. Shapiro. Weakly invertible elements in certain function spaces, and generators in ℓ_1. *Michigan Math. J.*, 11(1):161–165, 1964.

[18] H. S. Shapiro. A class of singular functions. *Canadian Journal of Mathematics*, 20:1425–1431, 1968.

[19] P. M. Tamrazov. Contour and solid structure properties of holomorphic functions of a complex variable. *Russian Mathematical Surveys*, 28(1):141, 1973.

Bartosz Malman,
KTH Royal Institute of Technology,
Stockholm, Sweden
malman@kth.se