Construction of some smooth Cauchy transforms

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Abstract

For a given Beurling-Carleson subset E of the unit circle $\mathbb{T}$ of has positive Lebesgue measure, we give explicit formulas for measurable functions supported on E such that their Cauchy transforms have smooth extensions to $\mathbb{T}$. The existence of such functions has been previously established by Khrushchev in 1978 in non-constructive ways by the use of duality arguments. We use our construction in two different applications. In the first application, we obtain an independent proof of a related principle of Khrushchev and Kegejan on simultaneous convergence of analytic polynomials in certain measures supported on the closed unit disk $\overline{\mathbb{D}}$. In the second application, for a rather large class of outer functions b which are extreme points of the unit ball of H^∞, we give a constructive algorithm for approximation by functions with smooth extensions to $\mathbb{T}$ in the extreme de Branges-Rovnyak spaces $\mathcal{H}(b)$. We give also a sufficient condition for an $\mathcal{H}(b)$-space to contain such a smooth function, and discuss limitations of our constructive approach.

1 Introduction

1.1 Background

Let E be a closed subset of the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ of the complex plane $\mathbb{C}$, and let the notation dm stand for the Lebesgue measure, normalized by the condition $m(\mathbb{T}) = 1$. The starting point for our development is the following question which has been studied and answered by Khrushchev in [14]. Namely, what conditions on the set E guarantee the existence of a non-zero measurable function h supported on E for which the

Cauchy transform, or *Cauchy integral*,

$$C_h(z) := \int_E \frac{h(\zeta)}{1 - z\bar{\zeta}} dm(\zeta), \quad z \in \mathbb{D}, \tag{1}$$

which is an analytic function in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, can be extended to a continuous function on the closed disk $\overline{\mathbb{D}}$? What conditions on E are necessary to assure existence of such a measurable function h for which also the complex derivative C'_h admits such an extension? By $\mathcal{A}$ we will denote the class of analytic functions in $\mathbb{D}$ which admit a continuous extension to $\overline{\mathbb{D}}$, and by $\mathcal{A}^n$ we denote those functions in $\mathcal{A}$ for which the n:th derivative admits such an extension.

We should think of the set E as being rather rough. Indeed, if E contains an arc A of the circle, then certainly any smooth function $s : \mathbb{T} \to \mathbb{C}$ with support on A will be transformed into a function C_s which is a member of $\mathcal{A}^\infty$ (which we define as the algebra of analytic functions in $\mathbb{D}$ for which derivatives of any order extend continuously to $\overline{\mathbb{D}}$, in other words $\mathcal{A}^\infty = \cap_n \mathcal{A}^n$). This follows readily from the rapid rate of decay of Fourier coefficients $\{s_n\}_n$ of the smooth function s, and the fact that $C_s(z) = \sum_{n=0}^{\infty} s_n z^n$.

Khrushchev in [14] has solved the existence part of the above problem in full. For a general closed set E, he establishes the existence of a non-zero measurable function h, with support only on E, such that C_h given by (1) is in the class $\mathcal{A}$. Moreover, he proves that a non-zero measurable function h supported on E for which the transform (1) is a function in $\mathcal{A}^\infty$ exists essentially if and only if E contains a so-called *Beurling-Carleson set* of positive measure. A set E is a Beurling-Carleson set if it is closed and if the following condition is satisfied:

$$\sum_{n=1} |A_n| \log(1/|A_n|) < \infty, \tag{2}$$

where $\{A_n\}_n$ is the disjoint union of open arcs of $\mathbb{T}$ which together equal the complement $\mathbb{T} \setminus E$, and $|A|$ is the length of the arc A. The class of Beurling-Carleson sets has a rich history, and appears notably in the solution of boundary zero set problems for smooth analytic functions and zero set problems for Bergman-like spaces (see Carleson's paper [3] and Korenblum's paper [16], for instance).

1.2 Khrushchev's methods and our main result

A notable feature of the proofs of the above mentioned results is that they are non-constructive. The existence of the measurable function h is established by duality arguments, and an explicit formula for h is lacking.

In the first case, when E is a general closed subset of $\mathbb{T}$, Khrushchev proves the existence of h by a duality argument involving the classical theorem of Khintchine–Ostrowski which deals with simultaneous convergence of Nevanlinna class functions on $\mathbb{D}$ and a subset of $\mathbb{T}$ (see [10] or [14] for a precise statement). In the second case, when E contains a Beurling-Carleson set of positive measure, he first proves a variant of the Khintchine-Ostrowski theorem for certain other classes of functions, and concludes the existence of h by a duality argument similar to the one in the first case. The mentioned variant of Khintchine-Ostrowski theorem has also been independently established by Kegejan in [13].

The main purpose of this article is to show that, at least in the second case in which Beurling-Carleson sets are involved, the theorem of Khrushchev can be obtained in a rather elementary and constructive way. Thus, in one of the main results, Proposition 3.1, we will give explicit and fully computable formulas for measurable functions h supported on a Beurling-Carleson set E for which C_h is a function in $\mathcal{A}^\infty$. We will outline the construction in Sections 2 and 3.

In fact, we will show an abundance of such functions h, plenty enough to derive several interesting consequences. We will give two main applications. In the first application, we will go in the direction reverse of the one taken by Khrushchev, and derive from our construction of smooth Cauchy transforms a version of the Khintchine-Ostrowski type result that Khrushchev used to prove his existence theorem. This is the content of Proposition 3.3, which we prove in Section 3, using a pure Hilbert space method in combination of the existence of smooth Cauchy transforms. In contrast, the method in [14] relies on estimates for harmonic measures for certain subdomains of the unit disk.

1.3 Constructive approximations in $\mathcal{H}(b)$-spaces

The second application concerns constructive approximations in a class of spaces of analytic functions called de Branges-Rovnyak spaces. We denote these by $\mathcal{H}(b)$. The symbol b is an analytic function $b : \mathbb{D} \to \mathbb{D}$, and the space $\mathcal{H}(b)$ space can be defined in terms of the symbol as the unique Hilbert space of analytic functions with reproducing kernel equal to

$$k_b(\lambda, z) = \frac{1 - \overline{b(\lambda)}b(z)}{1 - \overline{\lambda}z}, \quad \lambda, z \in \mathbb{D}. \tag{3}$$

In other words, $\mathcal{H}(b)$ is the closed linear span of the kernel functions $k_b(\lambda, \cdot)$, and the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}(b)}$ satisfies

$$\langle f, k_b(\lambda, \cdot) \rangle_{\mathcal{H}(b)} = f(\lambda), \quad \lambda \in \mathbb{D}. \tag{4}$$

There are several other ways in which one can define the space, but all the definitions express the space as the image of an operator (Toeplitz operator or Cauchy transform, for instance) or an orthogonal complement of a multiplication operator. As a consequence, for a general symbol b it can be quite difficult to tell what kind of functions are contained in $\mathcal{H}(b)$. See [7], [8] and [20] for the various way of constructing the space $\mathcal{H}(b)$, and for further background on de Branges-Rovnyak spaces.

The theory of $\mathcal{H}(b)$ spaces splits naturally into two very different cases. The first case, which we will call the *non-extreme* case, occurs when the defining symbol b satisfies

$$\int_{\mathbb{T}} \log(1 - |b(\zeta)|) dm(\zeta) > -\infty. \tag{5}$$

The above condition is equivalent to b being a non-extreme point of the unit ball of H^∞, which is the algebra of bounded analytic functions in $\mathbb{D}$. In this case, the space contains all analytic polynomials as a dense subset. The density of polynomials has been established by Sarason in the 1980s, and the proof method involved a duality argument. Much later, in 2016, a constructive proof of the density of polynomials has been obtained in [6], together with very interesting results on failure of various approximation techniques in $\mathcal{H}(b)$ spaces. Notably, and in contrast to most other well-studied separable spaces of analytic functions, it turns out that approximations by dilations $f_r(z) := f(rz), r \in (0,1)$ fail in general in $\mathcal{H}(b)$-spaces, and the dilation operators $f \mapsto f_r$ are not uniformly bounded as $r \to 1^+$ for certain symbols non-extreme symbols b.

The second case, the *extreme case*, occurs when the quantity in (5) equals minus infinity. With exception of the reproducing kernel functions given by (3), very few explicit formulas for inhabitants of extreme $\mathcal{H}(b)$ spaces are known, but it is known that the the set of analytic polynomials is never a subset. However, just as in the non-extreme case, duality approaches have proven to be very effective also in the extreme case in establishing the existence of large sets of functions in $\mathcal{H}(b)$ which exhibit some additional regularity, such as being members of $\mathcal{A}$ or $\mathcal{A}^n$. In particular, it has been established in [1] that the functions in $\mathcal{A} \cap \mathcal{H}(b)$ are always dense in $\mathcal{H}(b)$, while density of the more regular functions in the $\mathcal{A}^n$ classes requires some assumptions on the symbol b and is related to some other interesting questions in operator theory and spectral theory of function in the disk (see [18]).

In contrast to the fruitful duality approach, constructive approximation techniques in the extreme case of $\mathcal{H}(b)$ are lacking. As a consequence of our construction of smooth Cauchy transforms in Proposition 3.1, we are able to provide (to the author's best knowledge) the first constructive approximation algorithm which is applicable to certain $\mathcal{H}(b)$-spaces defined by extreme symbols b. We are far from being able to cover all symbols b, and the main simplifying assumption that we will employ is that we will consider outer symbols b only. On the upside, our approximation will be accomplished by smooth functions in the class $\mathcal{A}^\infty$. More precisely, given a function f in an $\mathcal{H}(b)$ space with an admissible outer symbol b, we will give explicit and computable formulas for functions $f_n \in \mathcal{H}(b) \cap \mathcal{A}^\infty$, and we will prove the convergence of the sequence $\{f_n\}_n$ to f in the $\mathcal{H}(b)$-norm.

Our goal is thus somewhat different than in the non-extreme constructive polynomial approximation scheme as presented in [6]. There, the point was to avoid completely the previous duality arguments, and to establish the density of polynomials "from scratch" by a purely constructive method. Naturally, given the knowledge of density of polynomials, regardless of how it was obtained, a constructive polynomial approximation scheme could be implemented by the usual Gram-Schmidt orthogonalization process. In the extreme case which we are considering here, a priori there exists no readily identifiable set of elements $\mathcal{H}(b) \cap \mathcal{A}^\infty$ to which any orthogonalization process could even be applied. In fact, establishing a large enough set of functions in $\mathcal{H}(b) \cap \mathcal{A}^\infty$ is our main difficulty.

The two algorithms deal with very different intial conditions, but it must be noted that the algorithm in [4] is more efficient than ours. Surprisingly, it implements the approximation without the need for an explicit formula for the so-called *mate* f_+ of a function $f \in \mathcal{H}(b)$ (see Section 4.1) which holds the information about the norm of f. The algorithm presented here will require an initial approximation by reproducing kernels, in the case of unknown mate f_+.

The algorithm is presented in Section 4 below. The exact assumptions we will put on b are the following:

(A) b is an outer function and an extreme point of the unit ball of H^∞,

(B) the set
$$E := \{\zeta \in \mathbb{T} : |b(\zeta)| < 1\}$$
is (up to a set of measure zero) a Beurling-Carleson set of positive measure which is not the whole circle,

(C) the weight $\Delta^2 := 1 - |b|^2$ is log-integrable on E:

$$\int_E \log(1 - |b|) dm > -\infty.$$

We emphasize that in (C), the integration domain is set E only, so the assumption does not make b a non-extreme point. For instance, the outer function b with modulus equal to 1 and $1/2$ on two arcs A_1 and A_2 respectively, with $A_1 \cup A_2 = \mathbb{T}$, satisfies all three of the above assumptions, but our class is of course much larger. Points (B) and (C) relate to assumptions in Proposition 3.1 below, which will be our main technical tool.

Out of the three assumptions, the first is restrictive, while the second and the third are somewhat easier to justify. Our algorithm implements an approximation by functions from the class $\mathcal{A}^\infty$, and from the duality theory developed in [18] it is known that some condition on structure of the set E above and on the size of the weight Δ are necessary for this kind of approximation to be possible. However, it is a fact that the approximation problem for outer symbols b is less complicated than the general case. We will exhibit below in 4.4 an example which shows just how intricate is the interplay between the inner and outer factors of b in the context of approximations by smooth functions, in spite of $\mathcal{H}(b)$ admitting very promising decompositions of the space into pieces coming from the inner and outer factors (see, for instance, [8, Theorem 18.7]).

2 Construction of an analytic cut-off function

We start off by presenting the constructing of a certain analytic function with strong decay properties near a given Beurling-Carleson set. The reason for calling it a *cut-off function*, as in name of the section, will become clear in the coming application in Proposition 3.1. Our construction is a straightforward adaptation of a technique from [11], more precisely from Lemma 7.11 of that work. We could have also followed the ideas of [19] or [22]. The proof is included for the reader's convenience.

Lemma 2.1. *Let E be a Beurling-Carleson set of positive measure. There exists an analytic function $g : \mathbb{D} \to \mathbb{D}$ such that the function $G(t) := g(e^{it})$ is smooth on $\mathbb{T} \setminus E$, and we have the estimate*

$$|G^{(m)}(t)| = o(dist(e^{it}, E)^N), \quad e^{it} \to E$$

for each pair of non-negative integer N and m. Here $G^{(m)}$ denotes the m:th derivative of G with respect to the variable t.

Proof. Let $\cup_{n\in\mathbb{N}} A_n = \mathbb{T} \setminus E$ be the complement of E with respect to $\mathbb{T}$. For each subarc A_n, we perform the classical *Whitney decomposition* $A_n = \cup_{k\in\mathbb{Z}} A_{n,k}$. More precisely, let $A_{n,0}$ be the arc with the same midpoint as A_n but having one third of the length of A_n. For this choice of the length we have $|A_{n,0}| = \operatorname{dist}(A_{n,0}, E)$. The arcs $A_{n,-1}$ and $A_{n,1}$ should be chosen adjacent to $A_{n,0}$ from the left and right respectively, and their lengths should be chosen, again, such that $|A_{n,-1}| = \operatorname{dist}(A_{n,-1}, E)$ and $|A_{n,1}| = \operatorname{dist}(A_{n,1}, E)$. It is easy to see that the correct choice is $|A_{n,1}| = |A_{n,-1}| = \frac{|A_n|}{6}$. Proceeding in this manner, we will obtain a decomposition

$$\mathbb{T} \setminus E = \cup_n A_n = \cup_{n,k} A_{n,k}$$

where for each arc $A_{n,k}$ we have

$$|A_{n,k}| = \frac{|A_n|}{3 \cdot 2^{|k|}} = \operatorname{dist}(A_{n,k}, E) \tag{6}$$

A straight-forward computation based on (6) will show that

$$\sum_{n,k} |A_{n,k}| \log(1/|A_{n,k}|) < \infty.$$

Let $\{B_j\}_j$ be a re-labelling of the arcs $\{A_{n,k}\}_{n,k}$ and $\{\lambda_j\}_j$ a positive sequence tending to infinity such that

$$\sum_j \lambda_j |B_j| \log(1/|B_j|) < \infty.$$

Now let $r_j = 1 + |B_j|$, $b_j \in \mathbb{T}$ be the midpoint of the arc B_j, and consider the function

$$h(z) = -\sum_j h_j(z) = -\sum_j \frac{\lambda_j b_j |B_j| \log(1/|B_j|)}{r_j b_j - z}, \quad z \in \mathbb{D}. \tag{7}$$

It is not hard to see that the real part of $h(z)$ is negative in $\mathbb{D}$. In fact, the real part of the j:th term in the sum is

$$-\operatorname{Re} h_j(z) = -\lambda_j |B_j| \log(1/|B_j|) \frac{\operatorname{Re}(r_j - \overline{z} b_j)}{|r_j b_j - z|^2} < 0,$$

where the last inequality follows from $\operatorname{Re}(r_j - \overline{z} b_j) < 0$, which is a consequence of the inequalities $r_j > 1$ and $|\overline{z} b_j| < 1$. It follows that

$$g(z) := \exp(h(z)) \tag{8}$$

is bounded by 1 in modulus for $z \in \mathbb{D}$. Moreover, the series defining $h(z)$ converges also for $z \in B_j$, and h_j extends analytically across each B_j, because the poles $\{r_j b_j\}_j$ of h cluster only at the set E. For $z \in B_j$, we have that the quantities $|r_j b_j - z|$ and $\operatorname{Re}(r_j - \overline{z} b_j)$ are both approximately equal to $|B_j|$, and so

$$|g(z)| \leq \exp(-\operatorname{Re} h_j(z)) \leq \exp(-c\lambda_j \log(1/|B_j|)) = |B_j|^{c\lambda_j}$$

for some positive constant c. Since $|B_j|$ equals the distance from B_j to E, for $z \in B_j$ we obtain

$$|g(z)| \leq C \operatorname{dist}(z, E)^{c\lambda_j}$$

for some positive constant $C > 0$ independent of j. Note that as z tends to E along the complement $\mathbb{T} \setminus E$, it needs to pass through infinitely many intervals B_j. Since λ_j tends to infinity, we obtain that

$$|g(z)| = o(\operatorname{dist}(z, E)^N) \tag{9}$$

as $z \to E$ along the complement of E on $\mathbb{T}$, for any choice of positive integer N.

Clearly $G(t) := g(e^{it})$ is smooth on $\mathbb{T} \setminus E$. On this set, the derivatives $G^{(m)}(t)$ have the form $H(e^{it})G(t)$, where H is a linear combination of products of derivatives of $h(e^{it})$ with respect to t. But a glance at (7) shows that such a product cannot grow faster than a constant multiple of $\operatorname{dist}(e^{it}, E)^{-n}$ for $e^{it} \in \mathbb{T} \setminus E$, for some integer n depending only on the number of derivatives taken. Together with (9), we see that the claim in the lemma follows. □

Note the fact that the proof above gives an explicit computable formula for the cut-off function g. It is given in terms of the Beurling-Carleson set E and is presented in equations (7) and (8).

3 Application I: A constructive proof of a theorem of Khrushchev

3.1 Smooth Cauchy transforms

Let E be a Beurling-Carleson set of positive measure. Lemma 2.1 will allow us to construct, and give explicit formulas for, measurable functions supported on E which have a smooth Cauchy transform. Thus we will now give the promised constructive proof of the theorem of Khrushchev from his seminal paper [14]. We state the theorem in the following somewhat more general form than it is stated in the mentioned work.

Proposition 3.1. ***(Construction of smooth Cauchy transforms)*** *Let E be a Beurling-Carleson set of positive measure such that $E \neq \mathbb{T}$, and w be a bounded positive measurable function with support on E which satisfies $\int_E \log(w)dm > -\infty$. Let W be the outer function*

$$W(z) = \exp\Big(\int_E \frac{z+\overline{\zeta}}{z-\overline{\zeta}} w(\zeta)dm(\zeta)\Big) \tag{10}$$

and g be the function associated to E which is given by Lemma 2.1. Consider the set

$$K = \Big\{ s = \overline{\zeta p g W} : p \text{ analytic polynomial} \Big\}. \tag{11}$$

Then the Cauchy transform

$$C_s(z) := \int_E \frac{s(\zeta)}{1 - z\overline{\zeta}} dm(\zeta)$$

is a non-zero function in $\mathcal{A}^\infty$ for each non-zero $s \in K$, the restrictions to E of elements of the set K form a dense subset of $L^2(1_E dm)$ and the set $CK = \{C_s : s \in K\}$ is dense in H^2.

Certainly our more general form of the theorem, together with the density statements, is obtainable by Khrushchev's methods in [14]. We therefore emphasize that our main contribution in this context are the explicit formulas for the measurable functions supported on E for which the Cauchy transform is an analytic function in $\mathcal{A}^\infty$. The formulas for the elements of K are given by the equations (7), (8) and (10).

The density statements in Proposition 3.1 will be useful for our further applications. It is not our point to prove these density statements constructively. In this part of the proof, we will use the following well-known theorem.

Lemma 3.2. ***(Beurling-Wiener theorem)*** *Let $M_{\overline{\zeta}} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ be the operator of multiplication by $\overline{\zeta}$. The closed $M_{\overline{\zeta}}$-invariant subspaces of $L^2(\mathbb{T})$ are of the form*

$$L^2(1_F dm) = \{f \in L^2(\mathbb{T}) : f = 0 \text{ almost everywhere on } \mathbb{T} \setminus F\}$$

where F is a measurable subset of $\mathbb{T}$, or of the form

$$U\overline{H^2} = \{U\overline{f} : f \in H^2\}$$

where U is a unimodular function.

For a proof of the Beurling-Wiener theorem, see [12], for instance.

Proof of Proposition 3.1. Since s is a conjugate analytic and satisfies $\int_{\mathbb{T}} s dm = 0$ we have

$$\int_{\mathbb{T}} \frac{s(\zeta)}{1 - z\overline{\zeta}} dm(\zeta) = 0$$

for each $z \in \mathbb{D}$. This implies that

$$C_s(z) = \int_E \frac{s(\zeta)}{1 - z\overline{\zeta}} dm(\zeta) = -\int_{\mathbb{T}\setminus E} \frac{s(\zeta)}{1 - z\overline{\zeta}} dm(\zeta). \tag{12}$$

Consider now the function $S(t) := s(e^{it})1_{\mathbb{T}\setminus E}(e^{it})$, where $1_{\mathbb{T}\setminus E}$ is the indicator function of the set $\mathbb{T} \setminus E$. From the formula (10) for W it is clear that this function extends analytically across $\mathbb{T} \setminus E$, and that the derivatives of W admit a bound $|W^{(m)})(e^{it})| \leq C \cdot \text{dist}(e^{it}, E)^{-2m}$ for $e^{it} \in \mathbb{T} \setminus E$. Thus derivative of any order of S tends to zero as e^{it} tends to E along $\mathbb{T} \setminus E$, and it is not hard to see that the derivatives of S vanish on E. Thus $S \in C^\infty(\mathbb{T})$. It follows that the Fourier coefficients S_n of S satisfy $|S_n| \leq C|n|^{-M}$ for some constant $C > 0$ and each positive integer M. Obviously then the function $C_s(z) = \sum_{n=0}^{\infty} S_n z^n$ is in $\mathcal{A}^\infty$. It is non-zero if s is non-zero, because the positive Fourier coefficients cannot vanish for the function S which is identically zero on an arc of $\mathbb{T}$.

The density in $L^2(1_E dm)$ of the restrictions to E of elements of the set K is an easy consequence of the invariance of K under multiplication by the coordinate function $\overline{\zeta}$ and the Beurling-Wiener theorem, Lemma 3.2 above. Indeed, the restriction to E of an element of K is non-zero almost everywhere on E, but obviously zero on $\mathbb{T} \setminus E$. It follows that the closure of K could not be anything else than $L^2(1_E dm)$.

The set CK is certainly contained in H^2, and the density in H^2 follows from the classical Beurling theorem for the Hardy spaces. More precisely, the set CK is invariant under the operator $f(z) \mapsto \frac{f(z)-f(0)}{z}$ (indeed, this operator takes $C_s \in CK$ to $C_{\overline{\zeta}s} \in CK$), and by Beurling's theorem the closure of CK is either all of H^2, or it coincides with a model space K_θ of functions which have boundary values on $\mathbb{T}$ of the form $\theta\overline{h}, h \in zH^2$, for some non-zero inner function θ. If we would be in the second case, then there would exist a function $k \in zH^2$ such that on the circle $\mathbb{T}$ we would have the equality $s1_E = C_s + \overline{k} = \theta\overline{h} + \overline{k}$, and consequently $\overline{\theta}s1_E \in \overline{H^2}$. This is a contradiction, since $\overline{\theta}s1_E$ vanishes on a set of positive measure. □

3.2 A principle of Khrushchev and Kegejan.

In Khrushchev's paper [14], Proposition 3.1 was derived by the use of duality from essentially the following statement which resembles the classical

Khintchine-Ostrowski theorem: if $E \subset \mathbb{T}$ is a Beurling-Carleson set of positive measure and $\{f_n\}_n$ is a sequence of analytic polynomials which satisfies the two assumptions

1. $\int_E |f_n| dm \to 0$ as $n \to \infty$, and
2. $|f_n(z)| \leq D(1-|z|)^{-C}$ for some positive constants C, D and $z \in \mathbb{D}$,

then we also have that

$$\lim_{n\to\infty} f_n(z) = 0, \quad z \in \mathbb{D},$$

uniformly on compact subsets of $\mathbb{D}$. Khrushchev actually proved his result in the more general context of λ-*Carleson* sets (see [14] for precise definition), and it was Kegejan in [13] who first established the result for Beurling-Carleson sets. One can formulate the above Khrushchev-Kegejan principle for Beurling-Carleson sets in terms of Hilbert spaces of functions in the following way.

Proposition 3.3. ***(Khrushchev-Kegejan principle, weak form)*** *Let $C > -1$ and E be a Beurling-Carleson set of positive measure. Let w be a bounded positive measurable function which is supported on E and satisfies $\int_E \log(w) dm > -\infty$. Consider the measure*

$$d\mu = (1-|z|^2)^C dA + w dm$$

and the classical Lebesgue space $L^2(\mu)$. Let $\mathcal{P}^2(\mu)$ be the closure of analytic polynomials in $L^2(\mu)$. If $f \in \mathcal{P}^2(\mu)$, then $f|\mathbb{D} \equiv 0$ if and only if $f|\mathbb{T} \equiv 0$.

In contrast to the method employed in [14], our proof of the existence of smooth Cauchy transforms in Proposition 3.1 is independent of Proposition 3.3. In fact, the two results are more or less equivalent, in the sense that we can also derive the latter from Proposition 3.1. We will now prove this by using a Hilbert space technique.

Proof of Proposition 3.3. The proof is very simple in principle. We will use the set K in (11), and this set will provide us with enough functionals to conclude that $f|\mathbb{D} \equiv 0$ and $f|\mathbb{T} \equiv 0$, respectively, by a straight-forward duality argument involving the Beurling-Wiener theorem.

Since the part of μ which lives on $\mathbb{D}$ is radial, we have

$$\int_{\mathbb{D}} f(z)\overline{g(z)} d\mu(z) = \sum_{k=0}^{\infty} f_k \overline{g_k} w_k(C),$$

for some positive numbers $w_k(C)$, and it is easy to verify the estimate $|w_k(C)| = O(n^{-C-1})$. The Taylor coefficients of functions in $CK \subset \mathcal{A}^\infty$ decay rapidly, and so it is easy to see that for each $s \in K$ there exists a function $G_s \in \mathcal{A}^\infty$ such that

$$\int_{\mathbb{D}} q(z)\overline{G_s(z)}(1-|z|^2)^C dA(z) + \int_E q(\zeta)\overline{s(\zeta)}dm(\zeta) = 0 \tag{13}$$

holds for all analytic polynomials p. In fact, all we need to do is choose

$$G_s(z) = \sum_{k=0}^{\infty} \frac{S_k}{w_k(C)} z^k,$$

where S_k is the k:th Fourier coefficient of the function $s1_E$ with $s \in K$. Note that (13) can be re-written as

$$\int_{\mathbb{D}} q(z)\overline{G_s(z)}(1-|z|^2)^C dA(z) + \int_E q(\zeta)\overline{\zeta p(\zeta) g(\zeta)}U(\zeta)w(\zeta)dm(\zeta) = 0 \tag{14}$$

for some unimodular function U.

Assume now that $f \in \mathcal{P}^2(\mu)$ is such that $f|\mathbb{T} \equiv 0$, and fix a sequence of analytic polynomials $\{p_n\}_n$ which tends to f in $L^2(\mu)$-norm. By (14), we have

$$\int_{\mathbb{D}} f(z)\overline{G_s(z)}(1-|z|^2)^C dA(z) = 0$$

for all $s \in K$. The restriction $f|\mathbb{D}$ is certainly a function in the Bergman space weighted by $(1-|z|^2)^C$, and the fact that $\{G_s\}_{s\in K}$ is dense in that space follows readily from the density of CK in H^2, which was established in Proposition 3.1. Thus $f|\mathbb{D} \equiv 0$.

Now assume that $f|\mathbb{D} \equiv 0$. Then similar reasoning as above leads us to the equality

$$\int_E f(\zeta)\overline{\zeta p(\zeta) g(\zeta)}U(\zeta)w(\zeta)dm(\zeta) = 0$$

for all analytic polynomials p. But as p runs through the analytic polynomials, the set $\{\overline{\zeta pg}U\}_p$ is dense in $L^2(wdm)$, again by the Beurling-Wiener theorem, just as in the proof of Proposition 3.1. Thus $f|\mathbb{T} \equiv 0$. □

A stronger result is reachable by the use of more sophisticated tools. In fact, one can strengthen the Khrushchev-Kegejan principle in the following way.

Proposition 3.4. ***(Khrushchev-Kegejan principle, strong form)*** *Let μ be as in Proposition 3.3. Then we have that $f \neq 0$ almost everywhere with respect to μ, for any non-zero $f \in \mathcal{P}^2(\mu)$.*

To derive the strong principle from the weak formulation, we can use a result of [2]. The weak principle, together with results of the cited work, imply that $f \in \mathcal{P}^2(\mu)$ has a non-tangential limit almost everywhere with respect to $\mu|\mathbb{T}$, and this limit equals $f|\mathbb{T}$. If f would vanish on a set of positive $\mu|\mathbb{T}$-measure, then a classical theorem of Privalov (see [15], for instance) can be used to deduce that $f \equiv 0$.

In other words, but somewhat imprecisely, the principles above states that a sequence of analytic polynomials cannot be small on $\mathbb{D}$ without being small on $\mathbb{T}$, as measured by μ. The above result falls under the theme of the uncertainty principle in function theory and harmonic analysis presented in [10]. Other manifestations of this principle are the already mentioned Khintchine-Ostrowski theorem, and the following classical statement in the theory of Hardy spaces: a function on the circle $\mathbb{T}$ with vanishing negative Fourier coefficients (i.e., having small spectrum) cannot be too small without being identically zero: $\int_{\mathbb{T}} \log |f| dm = -\infty$ if and only if $f \equiv 0$ for such functions. In fact we used this statement implicitly at the end of the proof of Proposition 3.1. We remark also that variants of Proposition 3.3 were used crucially in the development of duality approach to smooth approximations in $\mathcal{H}(b)$ spaces in the papers [18] and [17].

4 Application II: Constructive smooth approximations in some extreme $\mathcal{H}(b)$ spaces

In this section we will outline our algorithm for constructive approximation by smooth functions in extreme $\mathcal{H}(b)$-spaces where b satisfies the assumptions $(A), (B)$ and (C) given in Section 1.

4.1 Background on $\mathcal{H}(b)$-spaces.

Before going into the details of the algorithm, we should recall some facts about $\mathcal{H}(b)$-spaces which will be used in our development below. For further details, we refer to the works [20] and [7], [8]. In what follows, we deal strictly with the extreme case, and some of the claims apply to this case only.

We start by describing the norm in the space. Let T_h denote the usual Toeplitz operator with measurable symbol h:

$$T_a f(z) = \int_{\mathbb{T}} \frac{f(\zeta)h(\zeta)}{1 - z\overline{\zeta}} dm(\zeta).$$

A function f which is contained in the usual Hardy space H^2 is a member of $\mathcal{H}(b)$ if and only if the function $T_{\overline{b}}f$ can be realized as a Cauchy integral

$$T_{\overline{b}}f(z) = \int_E \frac{f_+(\zeta)}{1 - z\overline{\zeta}} \Delta(\zeta) dm, \tag{15}$$

where f_+ is a square-integrable measurable function living on the set $E = \{\zeta \in \mathbb{T} : |b(\zeta)| < 1\}$, and where $\Delta = \sqrt{1 - |b|^2}$. It is a fact that such a function f_+ is uniquely determined, and the norm in $\mathcal{H}(b)$ is given by

$$\|f\|^2_{\mathcal{H}(b)} = \|f\|^2_2 + \|f_+\|^2_2, \tag{16}$$

the expression $\| \cdot \|_2$ being the usual L^2-norm computed on the circle. See [1] for a derivation of this fact. We will call f_+ the *mate* of f. The pairs (f, f_+), and how different operations defined on $\mathcal{H}(b)$ translate into operations on these pairs, is central to the theory of $\mathcal{H}(b)$-spaces.

It is in general difficult to obtain the function f_+ from f. For the reproducing kernel functions given by (3) they can be computed with relative ease. We can verify directly that the mate of $k_b(\lambda, z)$ is

$$\frac{\overline{b(\lambda)}\Delta(\zeta)}{1 - \overline{\lambda}\zeta}, \quad \lambda \in \mathbb{D}, \zeta \in \mathbb{T}. \tag{17}$$

In case b is outer and satisfies $|b| = 1$ on an arc A of the unit circle, then the functions in $\mathcal{H}(b)$ admit analytic continuations to $\mathbb{D} \cup \mathbb{D}_e \cup A$, where $\mathbb{D}_e = \{z \in \mathbb{C} : |z| > 1\}$. This fact can be seen from the Aleksandrov-Clark representation formula for functions in $\mathcal{H}(b)$ (see [8, Theorem 20.5]). If the analytic continuations are known, then the mate f_+ of f can be computed by the use of Fatou's jump theorem:

$$\lim_{r \to 1} T_{\overline{b}}f(r\zeta) - T_{\overline{b}}f(\zeta/r) = i\Delta(\zeta)f_+(\zeta)$$

for almost every $\zeta \in \mathbb{T}$ (see [9], Exercise II.3). Here $T_{\overline{b}}f(\zeta/r)$ denotes the values of the analytic continuation of $T_{\overline{b}}f$ to $\mathbb{D} \cup \mathbb{D}_e \cup A$.

The function $T_{\overline{b}}f$ is itself a member of the space, as is $T_{\overline{h}}f$ for any bounded co-analytic symbol $\overline{h}$. Thus, $\mathcal{H}(b)$ is invariant for the co-analytic Toeplitz operators. The assignment $\overline{h} \to T_{\overline{h}}$ satisfies many nice properties. For instance, if $\{h_n\}_n$ is a sequence which converges boundedly to h on $\mathbb{T}$, then the operators $T_{\overline{h_n}}$ converge to $T_{\overline{h}}$ in the strong operator topology on $\mathcal{H}(b)$. This can be derived from the general properties of the Sz.-Nagy–Foias functional calculus (see [21]), but it can also be obtained as a quick corollary of the following fact which we will find use for below: if $f \in \mathcal{H}(b)$ and (f, f_+) is the corresponding pair as in (16), then the pair of $T_{\overline{h}}f$ is

$$(T_{\overline{h}}f, \overline{h}f_+). \tag{18}$$

One improvement of the convergence statement which we will find especially useful has been noted in [6].

Lemma 4.1. *If $\{h_n\}_n$ is a sequence of analytic functions bounded by 1, and $h_n(0) \to 1$ as $n \to \infty$, then the sequence of operators $\{T_{\overline{h_n}}\}_n$ converges to the identity in the strong operator topology on $\mathcal{H}(b)$:*

$$\lim_{n\to\infty} \|T_{\overline{h_n}} f - f\|_{\mathcal{H}(b)} = 0, \quad f \in \mathcal{H}(b). \tag{19}$$

This lemma has been fruitfully employed in [6], where it was an important tool in the constructive proof of the density of polynomials in the non-extreme $\mathcal{H}(b)$-spaces. The lemma will play a similar role in our development.

4.2 Description of the smooth approximation algorithm

4.2.1 Step 1: a preliminary approximation by non-smooth Cauchy integrals

In the first step, we will follow the same general idea as the constructive approximation scheme in [6], and we introduce the functions

$$M_n(z) = \exp\Big(\int_{\mathbb{T}} \frac{z+\overline{\zeta}}{z-\overline{\zeta}} \min(1, n|b(\zeta)|) dm(\zeta) \Big). \tag{20}$$

This is an outer function with modulus $|M_n(\zeta)| = \min(1, n|b(\zeta)|)$ on the circle $\mathbb{T}$. Importantly, we have

$$c_n := M_n/b \in H^\infty,$$

and in fact $\|c_n\|_\infty \leq n$. We have the factorization $M_n = c_n b$, and consequently we obtain the factorization $T_{\overline{M_n}} = T_{\overline{b}} T_{\overline{c_n}}$ for the corresponding co-analytic Toeplitz operators. If f is any function in $\mathcal{H}(b)$, then by the co-analytic Toeplitz operator invariance of $\mathcal{H}(b)$ we have that $T_{\overline{M_n}} f$ and $T_{\overline{c_n}} f$ are in $\mathcal{H}(b)$, and by the remark (18) on how these operators act on the pairs (f, g) and by equation (15), we have the representation formula

$$T_{\overline{M_n}} f(z) = T_{\overline{b}} T_{\overline{c_n}} f(z) = \int_E \frac{\overline{c_n(\zeta)} g(\zeta)}{1 - z\overline{\zeta}} \Delta(\zeta) dm(\zeta). \tag{21}$$

.

Note that $\min(1, n|b(\zeta)|)$ certainly tends to 1 almost everywhere on $\mathbb{T}$ as n tends to infinity, and thus by the dominated convergence theorem it follows from the equation (20) that $\lim_{n\to\infty} M_n(0) = 1$. Thus Lemma 4.1 ensures that

$$\lim_{n\to\infty} \|T_{\overline{M_n}} f - f\|_{\mathcal{H}(b)}. \tag{22}$$

We remark that in this step we have crucially used that b is outer. Indeed, if b had an inner factor, then the conclusion $c_n = M_n/b \in H^\infty$ above would be wrong, and consequently the factorization of Toeplitz operators which lead us to (15) would not be possible.

4.2.2 Step 2: a convergence argument

By Step 1, it suffices to now approximate functions of the form $T_{\overline{M_n}} f \in \mathcal{H}(b)$ with a function of class $\mathcal{A}^\infty$. The corresponding pair is $(T_{\overline{M_n}} f, \overline{M_n} f_+)$. We simplify the notation from Step 1 by letting M_n become M, and c_n we replace by $c := c_n = M/b$. Thus we deal with the pair

$$(T_{\overline{M}} f, \overline{M} f_+). \tag{23}$$

Assume that we can obtain a formula for a sequence of measurable functions $\{q_k\}_k$ supported on the set E (recall, this is the support of $1 - |b|^2 = \Delta^2$) such that

$$Q_k(z) := \int_E \frac{q_k(\zeta)}{1 - z\overline{\zeta}} \Delta(\zeta) dm \tag{24}$$

are functions in $\mathcal{A}^\infty$ and also

$$\lim_{k\to\infty} \|q_k - \overline{c} f_+\|_2 = 0. \tag{25}$$

Then certainly

$$\lim_{k\to\infty} \|\Delta q_k - \Delta \overline{c} f_+\|_2 = 0,$$

and since the Cauchy transform is a continuous operator from $L^2(\mathbb{T})$ to H^2, we would obtain from (25) and (21) the convergence

$$\lim_{k\to\infty} \|T_{\overline{M}} f - Q_k\|_2 = 0. \tag{26}$$

The functions in the sequence $\{Q_k\}_k$ are actually members of $\mathcal{H}(b)$. Indeed, their corresponding pairs are easily verified to be

$$(Q_k, \overline{b} q_k). \tag{27}$$

By (25), we have

$$\lim_{k\to\infty} \|\overline{b} q_k - \overline{M} f_+\|_2 = \lim_{k\to\infty} \|\overline{b} q_k - \overline{bc} f_+\|_2 = 0. \tag{28}$$

Putting it all together, we see that the norm formula (16) together with (23), (26), (27) and (28) imply that

$$\lim_{k\to\infty} \|T_{\overline{M}}f - Q_k\|_{\mathcal{H}(b)} = 0.$$

Hence, we need only to give formulas for the functions in the sequence $\{Q_k\}_k$, which we do in the next step.

4.2.3 Step 3: construction of a smooth approximating sequence

We will now give explicit computable formulas for functions in a sequence $\{q_k\}_k$ such that (25) holds and such that the function (24) is in $\mathcal{A}^\infty$. We will assume that the mate f_+ of f is known. We will deal with the contrary case in Step 4.

Let $w := \Delta, W, E$ and g be as in Proposition 3.1. Further, let $U = W/\Delta$, which is unimodular on E. The measurable function $\overline{c}f_+$ is a member of $L^2(1_E dm)$, and so by a similar argument as at the end of the proof of Proposition 3.1 it can be approximated in the L^2-norm on E by a sequence of functions of the form $\overline{\zeta p_n g U} 1_E$, where p_n is an analytic polynomial. This step can be done fully constructively, as long as we have a formula for $\overline{c}f_+$. Indeed, an orthonormal basis of $L^2(1_E dm)$ of the form $\{\overline{\zeta p_n g U} 1_E\}_n$, where p_n is a degree n analytic polynomial can be explicitly computed by employing the Gram-Schmidt orthogonalization process to the computable measurable functions $\{\overline{\zeta^{n+1} g U} 1_E\}_n$, and we can let q_k be the projection of $\overline{c}f_+$ onto the first k vectors in this basis. Then, because the functions q_k are linear combination of elements of the basis $\{\overline{\zeta p_n g U} 1_E\}_n$, the identity $\overline{U}\Delta = \overline{W}$ and Proposition 3.1 ensure that Cauchy transform (24) is in $\mathcal{A}^\infty$, and (25) holds also.

4.2.4 Step 4: dealing with uncomputable mate f_+

A formula for f_+ is critical for Step 3. If the mate f_+ of the given function $f \in \mathcal{H}(b)$ is not obtainable through other means, we can fix a sequence $\{k_b(\lambda_n, z)\}_n$ of reproducing kernels of $\mathcal{H}(b)$ which has dense linear span. A sequence of linear combinations of reproducing kernels converging to f in the norm of $\mathcal{H}(b)$ can be fully computed from the values of f (and without knowledge of f_+) by the reproducing property of the kernels and again by the use of Gram-Schmidt orthogonalization process applied to $\{k_b(\lambda_n, z)\}_n$. The reproducing kernel functions themselves can be approximated constructively by functions in $\mathcal{A}^\infty$ by following Steps 1 through 3, thanks to explicit formula (17) for the mate.

This finishes our description of the constructive approximation algorithm.

4.3 Existence of smooth functions in extreme $\mathcal{H}(b)$

Instead of density of smooth functions, one can instead ask for conditions on b which guarantee existence of at least one non-zero function in the intersection $\mathcal{H}(b) \cap \mathcal{A}^\infty$.

In the case of model spaces $K_\theta := \mathcal{H}(\theta)$, where the symbol θ is an inner function, it was Dyakonov and Khavinson who gave a necessary and sufficient condition for K_θ to contain a non-zero function in the class $\mathcal{A}^\infty$. The inner function θ factors as a product of a Blaschke factor B and a singular inner function S_ν, where ν is a positive singular Borel measure on $\mathbb{T}$, and it was shown in [5] that $K_\theta \cap \mathcal{A}^\infty$ will be non-trivial if and only if B is non-trivial or if $\nu(E) > 0$ for some Beurling-Carleson set E of measure zero.

We have not been able to find a necessary and sufficient condition for existence of non-zero $\mathcal{A}^\infty$ functions in $\mathcal{H}(b)$, and the problem does indeed seem to be complicated. Some evidence for this is given below in 4.4. Our above developments, and some previous results, let us however arrive at the following sufficient conditions.

Corollary 4.2. *Let $b = b_0 B S_\nu$ be a factorization of b into the outer factor b_0, Blaschke product B and the singular inner function S_ν. Assume that at least one of the following conditions holds:*

(i) *B vanishes at some point $\lambda \in \mathbb{D}$,*

(ii) *$\nu(E) > 0$ for some Beurling-Carleson set of measure zero,*

(iii) *$\int_E \log(1 - |b|) dm > -\infty$ for some Beurling-Carleson set of positive measure.*

Then $\mathcal{H}(b)$ contains a non-zero function in $\mathcal{A}^\infty$. In all these cases, there is an explicit integral formula for such a function.

Proof. The non-extreme case is covered by condition (iii) and $E = \mathbb{T}$. If one of the first two conditions hold, then the result follows by the theorem of Dyakonov and Khavinson from [5]. If the third condition holds and b is extreme, then we note that the outer function b_1 which has boundary values of modulus equal to $|b_0|$ on E and equal to 1 on $\mathbb{T} \backslash E$ satisfies the assumptions of our constructive smooth approximation algorithm. Moreover, it is easy to see that b/b_1 is a function in the unit ball of H^∞. This readily implies that $\mathcal{H}(b_1) \subset \mathcal{H}(b)$ (see, for instance, [8, Theorem 18.7]), and so $\mathcal{H}(b)$ contains non-zero functions in $\mathcal{A}^\infty$. $\square$

4.4 An important example

Because of various decomposition formulas for $\mathcal{H}(b)$ spaces in terms of factorizations of b into functions in the unit ball of H^∞ (see the already mentioned [8, Theorem 18.7]) one might wonder whether the constructive approximation can be in some way effectively reduced to studying separately the problem in the spaces $\mathcal{H}(b_0)$ and $\mathcal{H}(\theta)$, where b_0 and θ are the outer and inner factors of b, respectively. We have mentioned earlier that, in fact, this is not the case. The interference between the outer and the inner factor is very significant in the context of the approximation problem, and the following example illustrates this interference.

Proposition 4.3. *Let n be any positive integer and let $\mathcal{A}^n$ be the set of analytic functions in $\mathbb{D}$ with the n:th derivative extending continuously to the boundary of $\mathbb{D}$. There exists a singular inner function θ and an outer function b_0 such that*

$$\mathcal{H}(\theta) \cap \mathcal{A}^n = \{0\}$$

and

$$\mathcal{H}(b_0\theta) \cap \mathcal{A}^n$$

is dense in $\mathcal{H}(b_0\theta)$.

Proof. We let ν be any positive singular Borel measure which vanishes on Beurling-Carleson sets of measure zero and which moreover is supported on a proper closed subarc A of the unit circle $\mathbb{T}$. By the before mentioned result of Dyakonov and Khavinson, we will have $\mathcal{H}(\theta) \cap \mathcal{A}^n = \{0\}$ for $\theta = S_\nu$. Now let b_0 be the outer function which has boundary values of modulus 1/2 on A and 1 elsewhere on $\mathbb{T}$. The main theorem of [18] implies that $\mathcal{H}(b_0\theta) \cap \mathcal{A}^n$ is dense in $\mathcal{H}(b_0\theta)$. □

We remark that the proof of the density statement in the above proposition which appears in [18] is highly non-constructive. At the present time we do not know of a technique for constructive smooth approximations when such "bad" singular inner functions, as in the above proposition, appear as a factor in b.

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