Density of disc algebra functions in de Branges-Rovnyak spaces

Densité des fonctions dans l'algèbre du disque dans les espaces de de Branges-Rovnyak

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Abstract

We prove that functions analytic in the unit disk and continuous up to the boundary are dense in the de Branges-Rovnyak spaces induced by the extreme points of the unit ball of H^∞. Together with previous theorems it follows that this class of functions is dense in any de Branges-Rovnyak space.

Abstract

On démontre que les fonctions analytiques dans le disque unité et continues dans le disque fermé sont denses dans l'espace de Branges-Rovnyak généré par un point extrémal de la boule unité de H^∞. En utilisant aussi des théorèmes précédents il résulte que cette classe de fonctions est dense dans un espace de Branges-Rovnyak quelconque.

1 Introduction

Let H^∞ be the algebra of bounded analytic functions in the unit disk $\mathbb{D}$ in the complex plane, and denote by $\mathcal{A}$ the disc algebra, i.e. the subalgebra of H^∞ consisting of functions which extend continuously to the closed disk. The Hardy space H^2 consists of power series in $\mathbb{D}$ with square-summable coefficients. If $\mathbb{T}$ denotes the unit circle, we identify as usual H^2 with the closed subspace of $L^2(\mathbb{T})$ consisting of functions whose negative Fourier coefficients vanish. The orthogonal projection from $L^2(\mathbb{T})$ onto H^2 is denoted by P_+.

For $\phi \in L^\infty(\mathbb{T})$ let T_ϕ denote the Toeplitz operator on H^2 defined by $T_\phi f = P_+\phi f$. Given $b \in H^\infty$ with $\|b\|_\infty \leq 1$ we define the corresponding *de Branges-Rovnyak space* $\mathcal{H}(b)$ as

$$\mathcal{H}(b) = (1 - T_b T_{\overline{b}})^{1/2} H^2.$$

$\mathcal{H}(b)$ is endowed with the unique norm which makes the operator $(1 - T_b T_{\overline{b}})^{1/2}$ a partial isometry from H^2 onto $\mathcal{H}(b)$. Alternatively, $\mathcal{H}(b)$ is defined as the reproducing kernel Hilbert space with kernel

$$k_b(z, \lambda) = \frac{1 - \overline{b(\lambda)}b(z)}{1 - \overline{\lambda}z}.$$

$\mathcal{H}(b)$-spaces are naturally split into two classes with fairly different structures according to whether the quantity $\int_{\mathbb{T}} \log(1-|b|)\,dm$ is finite or not. Here m denotes the normalized arc-length measure on $\mathbb{T}$. The present note concerns the approximation of $\mathcal{H}(b)$-functions by functions in $\mathcal{A}\cap\mathcal{H}(b)$ and from the technical point of view there is a major difference between the two classes, which we shall briefly explain.

If $\int_{\mathbb{T}} \log(1-|b|)\,dm < \infty$, or equivalently, if b is a non-extreme point of the unit ball of H^∞, then $\mathcal{H}(b)$ contains all functions analytic in a neighborhood of the closed unit disk (see section (IV-6) of [9]). By a theorem of Sarason, the polynomials form a norm-dense subset of the space (see section (IV-3) of [9]). An interesting feature of the proofs of density of polynomials in an $\mathcal{H}(b)$-space is that the usual approach of approximating a function f first by its dilations $f_r(z) = f(rz)$, and then by their truncated Taylor series, or by their Cesàro means, does not work. Sarason's intial proof of density of polynomials is based on a duality argument. In recent years a more involved constructive polynomial approximation scheme has been obtained in [7].

The picture changes dramatically in the case when $\int_{\mathbb{T}} \log(1-|b|)\,dm = \infty$, or equivalently when b is an extreme point of the unit ball of H^∞. Then it is in general a difficult task to identify any functions in the space other than the reproducing kernels, and it might happen that $\mathcal{H}(b)$ contains no non-zero function analytic in a neighborhood of the closed disk. A special class of extreme points are the inner functions. If b is inner then $\mathcal{H}(b) = H^2 \ominus bH^2$ with equality of norms, and it is a consequence of a celebrated theorem of Aleksandrov [1] that in this case the intersection $\mathcal{A}\cap\mathcal{H}(b)$ is dense in the space. The result is surprising since, as pointed out above, in most cases it is not obvious at all that $\mathcal{H}(b)$ contains any non-zero function in the disk algebra $\mathcal{A}$.

Motivated by the situation described here, E. Fricain [5], raised the natural question whether Aleksandrov's result extends to all other $\mathcal{H}(b)$-spaces induced by extreme points b of the unit ball of H^∞. It is the purpose of this note to provide an affirmative answer to this question, contained in the main result below.

Theorem 1. *If b is an extreme point of the unit ball of H^∞, then $\mathcal{A}\cap\mathcal{H}(b)$ is a dense subset of $\mathcal{H}(b)$.*

Together with Sarason's result [9] on the density of polynomials in the non-extreme case, it follows that the intersection $\mathcal{A}\cap\mathcal{H}(b)$ is dense in the space $\mathcal{H}(b)$ for any b in the unit ball of H^∞. Our proof of Theorem 1 is deferred to Section 3 and relies on a duality argument. Therefore, just as the earlier proofs of Sarason and Aleksandrov, our approach is non-constructive. Section 2 serves to establish some preliminary results.

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2 Preliminaries

2.1 The norm on $\mathcal{H}(b)$. An essential step is the following useful representation of the norm in $\mathcal{H}(b)$. The authors have originally deduced the result using the techniques in [3] (see also [2, Chapter 3]), but once the goal is identified, several available techniques provide simpler proofs. For example, the proposition below can be deduced from results in [9]. For the sake of completeness, we include a new shorter proof.

Proposition 2. *Let b be an extreme point of the unit ball of H^∞ and let*

$$E = \{\zeta \in \mathbb{T} : |b(\zeta)| < 1\}.$$

Then for $f \in \mathcal{H}(b)$ the equation

$$P_+\bar{b}f = -P_+\sqrt{1-|b|^2}g.$$

has a unique solution $g \in L^2(E)$, and the map $J : \mathcal{H}(b) \to H^2 \oplus L^2(E)$ defined by

$$Jf = (f, g),$$

is an isometry. Moreover,

$$J(\mathcal{H}(b))^\perp = \Big\{(bh, \sqrt{1-|b|^2}h) : h \in H^2\Big\}.$$

Proof. Let

$$K = \Big\{(bh, \sqrt{1-|b|^2}h) : h \in H^2\Big\} \subset H^2 \oplus L^2(E)$$

and let P_1 be the projection from $H^2 \oplus L^2(E)$ onto the first coordinate H^2, i.e., $P_1(f, g) = f$. We observe first that $P_1|K^\perp$ is injective. Indeed, if $K^\perp$ contains a tuple of the form $(0, g) \in H^2 \oplus L^2(E)$, it follows that

$$\int_{\mathbb{T}} \bar{\zeta}^n g(\zeta)\sqrt{1-|b(\zeta)|^2}\, dm(\zeta) = 0,\ n \geq 0,$$

and consequently the function $g\sqrt{1-|b|^2}$ coincides a.e. with the boundary values of the complex conjugate of a function $f \in H^2_0$. But the assumption that b is an extreme point then implies that $\int_{\mathbb{T}} \log|f|\, dm = -\infty$, and since $f \in H^2$, we conclude that $f = 0$, i.e, $g = 0$. Thus, the space $\mathcal{H} = P_1K^\perp$ with the norm $\|f\|_{\mathcal{H}} = \|P_1^{-1}f\|_{H^2\oplus L^2(E)}$ is a Hilbert space of analytic functions on $\mathbb{D}$, contractively contained in H^2, in particular, it is a reproducing kernel Hilbert space. We now show that $\mathcal{H}$ equals $\mathcal{H}(b)$ by verifying that the reproducing kernels of the two spaces coincide. This follows from a simple computation. For $\lambda \in \mathbb{D}$, the tuple

$$\begin{aligned}(f_\lambda, g_\lambda) &= \Big(\frac{1-\overline{b(\lambda)}b(z)}{1-\bar{\lambda}z}, -\frac{\overline{b(\lambda)}\sqrt{1-|b(z)|^2}}{1-\bar{\lambda}z}\Big)\\ &= \Big(\frac{1}{1-\bar{\lambda}z}, 0\Big) - \Big(\frac{\overline{b(\lambda)}b(z)}{1-\bar{\lambda}z}, \frac{\overline{b(\lambda)}\sqrt{1-|b(z)|^2}}{1-\bar{\lambda}z}\Big)\end{aligned}$$

is obviously orthogonal to K, while the last tuple on the right hand side is in K, so that f_λ is the reproducing kernel in $\mathcal{H}$, which obviously equals the reproducing kernel in $\mathcal{H}(b)$. The first assertion in the statement is now self-explanatory. □

2.2 Cauchy transforms and two classical theorems. The dual $\mathcal{A}'$ of the disk algebra $\mathcal{A}$ can be identified with the space $\mathcal{C}$ of Cauchy transforms of finite Borel measures on $\mathbb{T}$. The Cauchy transform $C\mu$ of a measure μ is given by

$$C\mu(z) = \int_{\mathbb{T}} \frac{1}{1 - z\overline{\zeta}} d\mu(\zeta),$$

and the duality between $\mathcal{A}$ and $\mathcal{C}$ is given by the pairing

$$\langle f, C\mu \rangle = \lim_{r \to 1^-} \int_{\mathbb{T}} f(\zeta) \overline{C\mu(r\zeta)} dm(\zeta) = \int_{\mathbb{T}} f d\overline{\mu}.$$

A proof of this fact can be found, for example, in Section 4.2 of [6]. The space $\mathcal{C}$ is endowed with the obvious quotient norm and is continuously contained in all H^p spaces for $0 < p < 1$.

Recall that analytic functions f in $\mathbb{D}$ satisfy $\sup_{0<r<1} \int_{\mathbb{T}} \log^+ |f_r| dm < \infty$ if and only if they are quotients of H^∞-functions, in particular they have finite nontangential limits a.e. on $\mathbb{T}$ which define a boundary function denoted also by f. The class $N^+(\mathbb{D})$ consists of quotients of H^∞-functions such that the denominator can be chosen to be outer. It contains in particular all Hardy spaces $H^p, \ p > 0$.

Two classical theorems will play an important role in the proof of Theorem 1. The first is the following theorem of Vinogradov, which also plays a crucial role in the proof of Aleksandrov's result. A proof of the below theorem the can be found in [10].

Theorem 3. *Let $f \in \mathcal{C}$. If I is an inner function such that $f/I \in N^+(\mathbb{D})$, then $f/I \in \mathcal{C}$ and $\|f/I\| \leq \|f\|$.*

The second is the Khintchin-Ostrowski theorem, and it reads as follows. A proof can be found in Section 3.2 of [8].

Theorem 4. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions analytic in the unit disk satisfying the following conditions:*

(i) There exists a constant $C > 0$ such that for all $n \geq 1$ we have

$$\sup_{0<r<1} \int_{\mathbb{T}} \log^+(|f_n(r\zeta)|) dm(\zeta) \leq C.$$

(ii) On some set $E \subseteq \mathbb{T}$ of positive Lebesgue measure, the sequence f_n converges in measure to a function ϕ.

Then the sequence f_n converges uniformly on compact subsets of the unit disk to a function $f \in N^+(\mathbb{D})$, and moreover $f = \phi$ a.e. on E.

3 Proof of the main result

Due to Proposition 2 we can now implement Aleksandrov's strategy from [1] which will then be combined with Theorem 3 and Theorem 4. The following result extends Alexandrov's approach to the context of $\mathcal{H}(b)$-spaces, when b is extremal in the unit ball of H^∞.

Lemma 5. *Let $b : \mathbb{D} \to \mathbb{D}$ be analytic, $E = \{\zeta \in \mathbb{T} : |b(\zeta)| < 1\}$, $B = \mathcal{A} \oplus L^2(E)$ and $B' = \mathcal{C} \oplus L^2(E)$. Then the set*

$$S = \{(C\mu, h) : C\mu/b \in N^+(\mathbb{D}), C\mu/b = h/\sqrt{1 - |b|^2} \text{ a.e. on } E\}$$

is weak- closed in B'.*

Proof. Since $\mathcal{A} \oplus L^2(E)$ is separable and S is a linear subspace, it will be sufficient to show that S is weak-* sequentially closed (see Theorem 5, p.76 of [4]). Let $(C\mu_n, h_n)$ converge weak-* to $(C\mu, h)$, where $(C\mu_n, h_n) \in S$ for $n \geq 1$. Equivalently, $h_n \to h$ weakly in $L^2(E)$, and

$$\sup_n \|C\mu_n\| < \infty. \quad \lim_{n \to \infty} C\mu_n(z) = C\mu(z), \ z \in \mathbb{D}.$$

Now by passing to a subsequence and the Cesàro means of that subsequence we can assume that $h_n \to h$ in the L^2-norm. Finally, using another subsequence we may also assume that $h_n \to h$ pointwise a.e. on E. Let I_b be the inner factor of b. Since $C\mu_n/I_b \in N^+(\mathbb{D})$, it follows by Theorem 3 that $\{C\mu_n/I_b\}_{n=1}^\infty$ is a bounded sequence in $\mathcal{C}$ converging pointwise on $\mathbb{D}$ to $C\mu/I_b$. This implies weak-* convergence in $\mathcal{C}$, in particular, $C\mu/I_b \in \mathcal{C} \subset N^+(\mathbb{D})$, and consequently, $C\mu/b \in N^+(\mathbb{D})$. Moreover, we have a.e. on E that $C\mu_n/b = h_n/\sqrt{1 - |b|^2}$ which converges pointwise to $h/\sqrt{1 - |b|^2}$, hence we conclude that the sequence $C\mu_n$ converges in measure to some function ϕ on E. Fix any $p \in (0, 1)$. Then

$$\int_{\mathbb{T}} \log^+(|C\mu_n(r\zeta)|) dm(\zeta) \lesssim \int_{\mathbb{T}} |C\mu_n(r\zeta)|^p dm(\zeta) \lesssim \sup_n \|C\mu_n\|^p < \infty.$$

Thus the assumptions of Theorem 4 are satisfied, and so (a subsequence of) $C\mu_n$ converges a.e. on E to $C\mu$. This clearly implies $C\mu/b = h/\sqrt{1 - |b|^2}$ a.e. on E, i.e. $(C\mu, h) \in S$. □

We are now ready to complete the proof of the main theorem.

Proof of Theorem 1. Let J denote the embedding in Proposition 2. Based on the pairing described at the beginning of Section 2.2, a direct application of Proposition 2 gives

$$J(\mathcal{A} \cap \mathcal{H}(b)) = \cap_{h \in H^2} \ker l_h,$$

where the functionals l_h are identified with elements of $\mathcal{C} \oplus L^2(E)$ as

$$l_h = \Big(hb, h\sqrt{1 - |b|^2}\Big).$$

It is a consequence of the Hahn-Banach theorem that the annihilator $J(\mathcal{A} \cap \mathcal{H}(b))^\perp$ is the weak-* closure of the set of the functionals l_h. Since for all $h \in H^2$ we have $l_h \in S$, the set considered in Lemma 5, by the lemma we conclude that $J(\mathcal{A} \cap \mathcal{H}(b))^\perp \subset S$. Thus if $f \in \mathcal{H}(b)$ is orthogonal to $\mathcal{A} \cap \mathcal{H}(b)$, we must have $Jf \in S$, that is

$$Jf = (hb, h\sqrt{1 - |b|^2})$$

for some $h \in N^+(\mathbb{D})$. The boundary values of h satisfy

$$\int_{\mathbb{T}} |h|^2 dm = \int_{\mathbb{T}} |bh|^2 dm + \int_{\mathbb{T}} (1 - |b|^2)|h|^2 dm = \|f\|^2$$

and hence by the Smirnov maximum principle we have $h \in H^2$. But then by Proposition 2, $Jf \in J(\mathcal{H}(b))^\perp$, which gives $Jf = 0$ and the proof is complete. □

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