UNIVERSAL MULTIPLIERS FOR SUB-HARDY HILBERT SPACES

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Abstract. To every non-extreme point b of the unit ball of $\mathcal{H}^\infty$ of the unit disk there corresponds a Pythagorean mate, a bounded outer function a satisfying the equation $|a|^2 + |b|^2 = 1$ on the boundary of the disk. We study universal, i.e., simultaneous multipliers for families of de Branges-Rovnyak spaces $\mathcal{H}(b)$, and develop a general framework for this purpose. Our main results include a new proof of the Davis-McCarthy universal multiplier theorem for the class of all non-extreme spaces $\mathcal{H}(b)$, a characterization of the Lipschitz classes as the universal multipliers for spaces $\mathcal{H}(b)$ for which the quotient b/a is contained in a Hardy space, and a similar characterization of the Gevrey classes as the universal multipliers for spaces $\mathcal{H}(b)$ for which b/a is contained in a Privalov class.

1. Introduction

1.1. **De Branges-Rovnyak spaces.** The spaces $\mathcal{H}(b)$ were introduced by de Branges and Rovnyak in [4] and named after them. They form a family of Hilbert spaces of analytic functions on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ contained in the Hardy space $\mathcal{H}^2$ of square-summable coefficient power series in $\mathbb{D}$. The family is parametrized by symbols b in the unit ball of $\mathcal{H}^\infty$, the algebra of bounded analytic functions in $\mathbb{D}$, and a given symbol b defines uniquely the Hilbert space $\mathcal{H}(b)$ of functions with reproducing kernel of the form

$$k_b(z, \lambda) = \frac{1 - \overline{b(\lambda)}b(z)}{1 - \overline{\lambda}z}, \quad z, \lambda \in \mathbb{D}.$$

In general, it is not particularly easy to understand what functions are members of $\mathcal{H}(b)$. Various ways of constructing the space appear in the original work of de Branges and Rovnyak in [4], Sarason's short treatise in [24], and the more recent two-volume set [8], [9] by Fricain and Mashreghi. A variety of applications of spaces $\mathcal{H}(b)$ to operator theory and complex analysis are also treated in those works.

In this article, we study the multiplier algebras of spaces $\mathcal{H}(b)$:

$$\mathbf{Mult}(\mathcal{H}(b)) = \{m \in \mathcal{H}^\infty : mf \in \mathcal{H}(b) \text{ whenever } f \in \mathcal{H}(b)\}$$

The algebra $\mathbf{Mult}(\mathcal{H}(b))$ may very well be trivial. In the case that the symbol b is an inner function, then Crofoot observed in [2] that the only multipliers of $\mathcal{H}(b)$ are the constant functions. As usual, a function is *inner* if it is bounded in $\mathbb{D}$ and has boundary values on $\mathbb{T} = \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ of unit modulus almost everywhere. From works of Lotto and Sarason in [12] and [13] we know that plenty of non-constant multipliers exist whenever b is not inner. However, an explicit characterization of $\mathbf{Mult}(\mathcal{H}(b))$ exists only in a few very special cases. See our discussion in Section 1.4 below for some examples of results of this type.

2020 *Mathematics Subject Classification.* Primary 30H45; Secondary 47B32.
Key words and phrases. de Branges-Rovnyak spaces, multipliers.

We will concern ourselves only with the *non-extreme case*, namely, we are assuming that the symbol b satisfies the condition

$$\int_{\mathbb{T}} \log(1 - |b(\zeta)|^2)|d\zeta| > -\infty.$$

Here $|d\zeta|$ denotes the arclength measure on $\mathbb{T}$. The above integral convergence is well known to be equivalent to b being a non-extreme point of the unit ball of $\mathcal{H}^\infty$. It is customary to introduce the outer function $a : \mathbb{D} \to \mathbb{D}$ satisfying the equation

$$|a(\zeta)|^2 + |b(\zeta)|^2 = 1$$

for almost every $\zeta \in \mathbb{T}$, and $a(0) > 0$. Then a is uniquely determined by b, and we say that b and a form a *Pythagorean pair*. These pairs will play a lead role in our discussion.

One can define a non-extreme space in a particularly useful way as a set of solutions to the *mate equation*,

$$\mathcal{T}_{\overline{b}} f = \mathcal{T}_{\overline{a}} f_+. \tag{1.1}$$

Here $\mathcal{T}_{\overline{b}}$ and $\mathcal{T}_{\overline{a}}$ are Toeplitz operators on $\mathcal{H}^2$, that is, operators of multiplication by $\overline{b}$ and $\overline{a}$ respectively, followed by an orthogonal projection from $L^2(\mathbb{T})$ to $\mathcal{H}^2$. The space $\mathcal{H}(b)$ can be defined as the set of those $f \in \mathcal{H}^2$ for which a solution $f_+ \in \mathcal{H}^2$ exists to the mate equation. It can be shown that at most one solution f_+ exists in $\mathcal{H}^2$, so that the *mate* f_+ is well-defined, if it exists. The norm on $\mathcal{H}(b)$ is then $\|f\|_{\mathcal{H}(b)} = \sqrt{\|f\|_2^2 + \|f_+\|_2^2}$, where $\|\cdot\|_2$ is the usual $L^2(\mathbb{T})$ norm. We note also that the non-extreme case is characterized by the containment of the set of analytic polynomials in $\mathbf{Mult}(\mathcal{H}(b))$, and their density in $\mathcal{H}(b)$. Both of these results are due to Sarason.

1.2. **Universal multipliers and the Davis-McCarthy theorem.** One special feature of the spaces $\mathcal{H}(b)$ is that they are not rotationally invariant. Namely, if $f(z) \in \mathcal{H}(b)$, then it is not in general the case that $f(e^{i\theta}z) \in \mathcal{H}(b)$. This feature is shared by the multiplier algebra $\mathbf{Mult}(\mathcal{H}(b))$, and membership of a function f in $\mathcal{H}(b)$ or in $\mathbf{Mult}(\mathcal{H}(b))$ often depends on local behaviour of f near distinguished points on $\mathbb{T}$ (for instance, this is the case in examples studied in [7]).

Consider, however, a family $\mathcal{F}$ of symbols b which is rotationally invariant: $b(e^{i\theta}z) \in \mathcal{F}$ for every $b(z) \in \mathcal{F}$ and $\theta \in \mathbb{R}$. Then certainly it is to be expected that $\cap_{b\in\mathcal{F}}\mathbf{Mult}(\mathcal{H}(b))$ is a space which is at least rotationally invariant, and one may therefore hope for an easier characterization of this intersection. Any function m inside the intersection of the multiplier algebras corresponding to $\mathcal{F}$ may justly be called a *universal multiplier* for $\mathcal{F}$. Our families of symbols b will be defined by membership of the *Pythagorean quotient* b/a in a sufficiently nice space of analytic functions X:

$$\mathcal{F}(X) := \{b : b/a \in X\}.$$

For instance, if $\mathcal{N}^+$ is the Smirnov class of quotients of bounded analytic functions in $\mathbb{D}$ with outer denominator, then $\mathcal{F}(\mathcal{N}^+)$ is readily seen to equal the family of all non-extreme symbols b. The universal multipliers in this case have been characterized by Davis and McCarthy in their deep work [3]. For $\alpha \in (0, 1/2]$, let $\mathcal{G}_\alpha$ be the Gevrey class

$$\mathcal{G}_\alpha := \Big\{ f(z) = \sum_{n\geq 0} \widehat{f}(n) z^n : |\widehat{f}(n)| = O\big(\exp(-cn^\alpha)\big) \text{ for some } c > 0 \Big\}. \tag{1.2}$$

Using McCarthy's earlier work on topologies of $\mathcal{N}^+$ in [15], they proved the following result.

Theorem (Davis-McCarthy). *We have*

$$\mathcal{G}_{1/2} = \bigcap_{b \in \mathcal{F}(\mathcal{N}^+)} \boldsymbol{Mult}(\mathcal{H}(b)).$$

Motivated by the above theorem, we will consider in this article smaller rotationally invariant families $\mathcal{F}(X)$, and characterize their corresponding universal multipliers.

1.3. **Main result.** Our first main result pertains to symbol families with Pythagorean quotients in $\mathcal{H}^p$. The Hardy space $\mathcal{H}^p$ consists of those analytic functions in $\mathbb{D}$ which satisfy the integral mean boundedness condition

$$\|f\|_p^p := \sup_{r \in (0,1)} \int_{\mathbb{T}} |f(r\zeta)|^p |d\zeta| < \infty. \tag{1.3}$$

We will use the full range $p \in (0, \infty)$. For $\alpha \in (0,1)$, the analytic Lipschitz class Λ^a_α is defined as the set of those functions m analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}} := \mathbb{D} \cup \mathbb{T}$ which satisfy the Lipschitz-type modulus of continuity estimate

$$|m(z) - m(w)| \leq C_m |z - w|^\alpha, \quad z, w \in \mathbb{D}$$

for some constant $C_m > 0$. For $\alpha = 1$, in order to present a unified statement of our theorem, we follow a usual convention: we define Λ^a_1 as the analytic Zygmund class consisting of functions m analytic in $\mathbb{D}$, continuous in $\overline{\mathbb{D}}$, and satisfying

$$|m(e^{i(t+s)}) + m(e^{i(t-s)}) - 2m(e^{it})| \leq C_m |s|, \quad s, t \in \mathbb{R}.$$

For $\alpha > 1$, we define Λ^a_α as the space of functions for which the appropriate derivative lies in one of the above introduced classes. Namely, if n is the positive integer satisfying $n < \alpha \leq n+1$, then Λ^a_α is to consist of those functions for which the derivative $f^{(n)}$ lies in $\Lambda^a_{\alpha-n}$. With these definitions in place, we can state our first main theorem.

Theorem A. *For $p \in (0, \infty)$ we have*

$$\Lambda^a_{1/p} = \bigcap_{b \in \mathcal{F}(\mathcal{H}^p)} \boldsymbol{Mult}(\mathcal{H}(b)).$$

One immediate corollary is that

$$\mathcal{A}^\infty = \bigcap_{p \in (0,\infty)} \bigcap_{b \in \mathcal{F}(\mathcal{H}^p)} \mathbf{Mult}(\mathcal{H}(b)),$$

where $\mathcal{A}^\infty = \bigcap_{\alpha > 0} \Lambda^a_\alpha$ is the algebra of analytic functions in $\mathbb{D}$ with smooth extensions to $\mathbb{T}$.

Our second result pertains to the Privalov classes $\mathcal{N}^q$. Recall that functions in $\mathcal{N}^+$ satisfy $\int_{\mathbb{T}} \log(1 + |f|)|d\zeta| < \infty$. For $q > 1$, the class $\mathcal{N}^q$ consists of those functions $f \in \mathcal{N}^+$ for which we have

$$\int_{\mathbb{T}} \big(\log(1 + |f|)\big)^q |d\zeta| < \infty.$$

The classes $\mathcal{N}^q$ have been studied by Privalov in [20].

Recall the definition of the Gevrey classes $\mathcal{G}_a$ in (1.2). Our second main result is the following.

Theorem B. *For $q \in (1, \infty)$ we have*

$$\mathcal{G}_{1/(1+q)} = \bigcap_{b \in \mathcal{F}(\mathcal{N}^q)} \boldsymbol{Mult}(\mathcal{H}(b)).$$

In the proof of our theorems we will use mainly the methods of functional analysis, and our approach focused around the mate equation in (1.1) is much different from the one used by Davis and McCarthy in [3]. The main results, Theorem A and Theorem B, will be deduced as consequences of a more general statement in Proposition 5.1 below. Given a space X satisfying certain properties, we will establish in that proposition a bijection between universal multipliers for $\mathcal{F}(X)$ and symbols of certain Hankel operators. In particular, our general result applies to $X = \mathcal{N}^+$, and so we will obtain a new proof of the Davis-McCarthy theorem, one which is independent from McCarthy's work on topologies of the Smirnov class from [15]. Our starting point will be the research of Lotto and Sarason from [13] which deals with operator-theoretic characterization of multipliers of $\mathcal{H}(b)$ in terms of boundedness of compositions of Hankel operators. An exposition of their results is contained in Section 2 below.

1.4. **Other multiplier results and comments.**

1.4.1. *More on the Davis-McCarthy theorem and its variants.* The original proof of Davis-McCarthy universal multiplier theorem in [3] is based on earlier results of McCarthy from [15] on the dual of $\mathcal{N}^+$. There is a natural metric on $\mathcal{N}^+$ (see Section 3.2 below), and the dual with respect to the induced metric topology has been found by Yanagihara in [26] to equal the Gevrey class $\mathcal{G}_{1/2}$ in (1.2). The dual of $\mathcal{N}^+$ with respect to a different topology, the so-called Helson topology, is easily seen to equal the intersection of ranges of all co-analytic Toeplitz operators on $\mathcal{H}^2$ (see [15] for details). McCarthy showed that the two mentioned topologies have the same duals, and concluded that $\mathcal{G}_{1/2}$ equals the intersection of ranges of all co-analytic Toeplitz operators. The relevance of this result to spaces $\mathcal{H}(b)$ is that any multiplier for $\mathcal{H}(b)$ must necessarily be of the form $m = \mathcal{T}_{\overline{a}}u$ for some $u \in \mathcal{H}^2$ (see [13], for instance), from which it easily follows that a universal multiplier must be contained in the mentioned intersection. That is, it necessarily must be a member of $\mathcal{G}_{1/2}$, by McCarthy's theorem.

Other authors proved variants of the Davis-McCarthy theorem by following their strategy outlined in the above paragraph. In [18], Meštrović and Pavićević used earlier duality results of Stoll from [25] to find the universal multipliers corresponding to the family of spaces $\mathcal{H}(b)$ for which the logarithm of the density of the Aleksandrov-Clark measure of b is in $L^q(\mathbb{T})$, $q > 1$. The condition to be a universal multiplier for this family is the same as in Theorem B, but the two results are different because the families of $\mathcal{H}(b)$-symbols are different.

Similarly, our proof of Theorem A is based on Lemma 3.2 below, which is a result of Duren, Romberg and Shields, but in fact going back to Hardy and Littlewood, and which identifies the dual space of $\mathcal{H}^q$, for $q \in (0, 1)$, as a Lipschitz class. Note, however, that in spite of this similarity, the proof technique used in the present article is completely different from the one used in [3] and the related works. One advantage of our approach is that it allows us to compute the universal multipliers for families of symbols b defined in terms of their modulus alone, instead of the less tractable logarithmic integrability of their Aleksandrov-Clark densities.

1.4.2. *Explicit characterization of multipliers for rational b.* Dealing with a single symbol b is a different and more delicate question, but an explicit characterization of $\mathbf{Mult}(\mathcal{H}(b))$ exists in the case that b is a rational function. In the case that a rational b is not inner, then it is well known that $\mathcal{H}(b) = M(\overline{a})$, where the latter space is the range of the coanalytic Toeplitz operator $\mathcal{T}_{\overline{a}}$. By a result of Fricain, Hartmann and Ross from [6] we conclude that $\mathbf{Mult}(\mathcal{H}(b)) = \mathcal{H}(b) \cap \mathcal{H}^\infty$ if b is rational and not inner. They establish also a very concrete description of the members of $\mathcal{H}(b)$ for these rational b. Naturally, for a general symbol b, such a simple characterization of $\mathbf{Mult}(\mathcal{H}(b))$ is not available.

1.4.3. *All bounded functions as multipliers.* The case $\mathcal{H}^\infty = \mathbf{Mult}(\mathcal{H}(b))$ has been settled by Sarason in [23, Theorem 3], who gives several conditions equivalent to this equality. One of the equivalent conditons is that $\mathcal{H}(b) = a\mathcal{H}^2 = \{af : f \in \mathcal{H}^2\}$. Another is that a and b should form a *Corona pair*, in the sense that

$$\inf_{z\in\mathbb{D}} (|a(z)| + |b(z)|) > 0$$

and $|a|^2$ should be a so-called A_2-weight: namely, it should satisfy the estimate

$$\left(\int_I |a|^2 |d\zeta| \right) \left(\int_I |a|^{-2} |d\zeta| \right) \leq C|I|^2,$$

where I is any arc of $\mathbb{T}$ and $C > 0$ is a constant. Davis and McCarthy in [3] found a similar characterization in terms of the density $w := (1 - |b|^2)|1 - b|^{-1}$ of the Aleksandrov-Clark measure of b. By their result, $\mathbf{Mult}(\mathcal{H}(b)) = \mathcal{H}^\infty$ if and only if w is an A_2-weight.

1.5. **Outline of the paper and the methods.** We begin, in Section 2, by presenting the results of Lotto and Sarason from [13]. Then, in Section 3, we characterize the continuity of Hankel operators between relevant pairs of function spaces. We continue in Section 4, by proving a stability result for Pythagorean factorizations of functions in $\mathcal{N}^+$. We make use of these results in Section 5, where we prove our general universal multiplier criterion and deduce from it Theorem *A* and Theorem *B*, as well as the Davis-McCarthy theorem. Then we conclude in Section 6 with some suggestions for continuing research along this direction.

2. Research of Lotto and Sarason

The purpose of this section is to present the research of Lotto and Sarason from [13]. The important consequences of the results from that work are stated in Corollary 2.2 and Corollary 2.4 below.

2.1. **Hankel operators.** Let P_+ and P_- be the standard projection operators

$$P_+ f(z) = \int_{\mathbb{T}} \frac{f(\zeta)}{1 - \overline{\zeta} z} |d\zeta|, \quad z \in \mathbb{D},\ f \in L^1(\mathbb{T}), \tag{2.1}$$

$$P_- f(z) = \int_{\mathbb{T}} \frac{f(\zeta)\zeta\overline{z}}{1 - \zeta\overline{z}} |d\zeta|, \quad z \in \mathbb{D},\ f \in L^1(\mathbb{T}). \tag{2.2}$$

The function $P_+ f(z)$ is analytic in $\mathbb{D}$. On the other hand, $P_- f(z)$ is conjugate analytic in $\mathbb{D}$, and it vanishes at $z = 0$. Through the usual identification of the Hardy space $\mathcal{H}^2$ with a closed subspace of $L^2(\mathbb{T})$, as well as similar identification of the orthogonal complement $L^2(\mathbb{T}) \ominus \mathcal{H}^2 = \overline{z\mathcal{H}^2} = \{\overline{zf} : f \in \mathcal{H}^2\}$, the operators P_+ and P_- are the orthogonal projections from $L^2(\mathbb{T})$ onto $\mathcal{H}^2$ and $\overline{z\mathcal{H}^2}$, respectively. The actions of P_+ and P_- on a Fourier series $f(\zeta) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\zeta^n \in L^2(\mathbb{T})$ are given by

$$P_+ f(\zeta) = \sum_{n\geq 0} \widehat{f}(n)\zeta^n$$

and

$$P_- f(\zeta) = \sum_{n<0} \widehat{f}(n)\zeta^n.$$

We highlight the easily established identity

$$\overline{P_- f} = P_+\overline{f} - \overline{\widehat{f}(0)}. \tag{2.3}$$

Given a symbol $m \in L^2(\mathbb{T})$, we will consider the *Hankel operator*

$$H_m f := P_- m f, \quad f \in \mathcal{H}^\infty. \tag{2.4}$$

Clearly $H_m f(z)$ is a conjugate analytic function in $\mathbb{D}$ which vanishes at $z = 0$. In particular, we have $H_m f \in \overline{z\mathcal{H}^2}$, and the conjugate $\overline{H_m f}$ is a member of the space $\mathcal{H}^2$.

The content of Nehari's theorem from [19] is that if $m \in \mathcal{H}^2$ equals the projection $m = P_+ u$ for some $u \in L^\infty(\mathbb{T})$, then the operator $H_{\overline{m}}$ acts boundedly from $\mathcal{H}^2$ into $\overline{z\mathcal{H}^2}$, and that this condition on m is necessary for boundedness. The space of such symbols, namely

$$\mathbf{BMOA} = \left\{ m \in \mathcal{H}^2 : m = P_+ u \text{ for some u } \in L^\infty(\mathbb{T}) \right\}$$

is the space of analytic functions of *bounded mean oscillation*. It is well known that $\mathcal{H}^1$ and **BMOA** are dual to each other, in the sense that every bounded linear functional ℓ on $\mathcal{H}^1$ can be identified with a unique element $m \in \mathbf{BMOA}$ for which we have

$$\ell(f) = \lim_{r\to 1-} \int_{\mathbb{T}} f(r\zeta)\overline{m(r\zeta)}|d\zeta| \tag{2.5}$$

The operator norm of the functional ℓ is comparable to $\|m\|_{\mathbf{BMOA}} := \inf \|u\|_\infty$, this infimum extending over all bounded functions u satisfying $m = P_+ u$. We say that **BMOA** is the *Cauchy dual* of $\mathcal{H}^1$.

2.2. **Lotto-Sarason characterization.** In [13], Lotto and Sarason characterized the membership $m \in \mathbf{Mult}(\mathcal{H}(b))$ in terms of boundedness of compositions of Hankel operators and their adjoints. The formal adjoint of the operator H_m is given by

$$H_m^* g = P_+ \overline{m} g$$

where $g \in \overline{z\mathcal{H}^2} \cap \overline{\mathcal{H}^\infty}$, say. Then H_m extends to a bounded operator $\mathcal{H}^2 \to \overline{z\mathcal{H}^2}$ if and only if H_m^* extends to a bounded operator $\overline{z\mathcal{H}^2} \to \mathcal{H}^2$, and if this is the case, these operators are each others Hilbert space adjoints. The following result (see [13, Theorem 2] for a proof) characterizes multipliers for $\mathcal{H}(b)$:

Theorem 2.1. *Let b be a non-extreme point of the unit ball of $\mathcal{H}^\infty$, and let a be the Pythagorean mate of b. A function $m \in \mathcal{H}^\infty$ is a member of $\mathbf{Mult}(\mathcal{H}(b))$ if and only if the following three conditions hold.*

(i) There exists $u \in \mathcal{H}^2$ such that $m = \mathcal{T}_{\overline{a}} u$.
(ii) The operator $H_{\overline{u}}^* H_{\overline{a}}$ is bounded on $\mathcal{H}^2$.
(iii) The operator $H_{\overline{u}}^* H_{\overline{b}}$ is bounded on $\mathcal{H}^2$.

Note that the conditions (ii) and (iii) may be satisfied even without the operator $H^*_{\overline{u}}$ being bounded by itself. The question of when a composition of Hankel operators as above is bounded is rather delicate. Theorem 2.1 gives us, however, a sufficient condition for membership of $m \in \mathbf{Mult}(\mathcal{H}(b))$. Namely, if the solution u to the equation $m = \mathcal{T}_{\overline{a}}u$ happens to satisfy $u \in \mathbf{BMOA}$, then the operator $H_{\overline{u}}$ is bounded by Nehari's theorem, and hence conditions (ii) and (iii) in Theorem 2.1 are satisfied by virtue of $a, b \in \mathcal{H}^\infty \subset \mathbf{BMOA}$. This simple consequence of Theorem 2.1 will play an important role in our development.

Corollary 2.2. *If $m \in \mathcal{H}^\infty$ is of the form*

$$m = \mathcal{T}_{\overline{a}}u$$

*for $u \in$ **BMOA**, then $m \in$ **Mult**$(\mathcal{H}(b))$.*

Lotto and Sarason in [13] used Corollary 2.2 to give new proofs of certain known statements regarding multipliers on $\mathcal{H}(b)$. They proved also a remarkable theorem characterizing when a multiplier $m \in \mathbf{Mult}(\mathcal{H}(b))$ is simultaneously a multiplier for the family of spaces $\{\mathcal{H}(Ib)\}_I$, where I is any inner function.

Theorem 2.3. *Let $m = \mathcal{T}_{\overline{a}}u \in$ **Mult**$(\mathcal{H}(b))$. The following two statements are equivalent.*

(i) We have

$$m \in \bigcap_I \boldsymbol{Mult}(\mathcal{H}(Ib)),$$

where the intersection is taken over all inner functions I.

(ii) The Hankel operator $H_{\overline{u}b} : \mathcal{H}^2 \to \overline{z\mathcal{H}^2}$ is bounded.

The easily verified identity $H_{\overline{u}b} = H_{P_-\overline{u}b}$ and Nehari's theorem imply that the second condition in Theorem 2.3 is equivalent to $\overline{P_-\overline{u}b}$ being a member of **BMOA**. In fact, one can express the condition (ii) intrinsically in terms of the space $\mathcal{H}(b)$. Indeed, condition (ii) is equivalent to the mate m_+ of m (recall the mate equation in (1.1)) being contained in **BMOA**, as follows from the proof of the following corollary.

Corollary 2.4. *Let $\mathcal{F}$ be a family of symbols invariant under multiplication by inner functions: $b \in \mathcal{F}$ implies that $Ib \in \mathcal{F}$ for every inner function I. If m is a universal multiplier for $\mathcal{F}$, then for any $b \in \mathcal{F}$, the mate m_+ of m in $\mathcal{H}(b)$ is a member of **BMOA**.*

Proof. Since m is contained in the intersection of the algebras $\mathbf{Mult}(\mathcal{H}(b))$ for $b \in \mathcal{F}$, in particular it is contained in the intersection of the algebras $\mathbf{Mult}(\mathcal{H}(Ib))$, where I is an arbitrary inner function. By Theorem 2.3, the operator $H_{\overline{u}b} : \mathcal{H}^2 \to \overline{z\mathcal{H}^2}$ is bounded, where $u \in \mathcal{H}^2$ satisfies $m = \mathcal{T}_{\overline{a}}u$. By Nehari's theorem, $P_-(\overline{u}b) \in \overline{\mathbf{BMOA}}$, the space of complex conjugates of functions in **BMOA**. Relation (2.3) implies that $P_+(\overline{b}u) = \mathcal{T}_{\overline{b}}u \in \mathbf{BMOA}$. Now, note that by the commutation relation stated in (5.2) below we have

$$\mathcal{T}_{\overline{a}}\mathcal{T}_{\overline{b}}u = \mathcal{T}_{\overline{b}}\mathcal{T}_{\overline{a}}u = \mathcal{T}_{\overline{b}}m.$$

This identity and (1.1) tells us that $\mathcal{T}_{\overline{b}}u$ is the mate of m in $\mathcal{H}(b)$. That is, $m_+ = \mathcal{T}_{\overline{b}}u \in \mathbf{BMOA}$. □

3. Hankel operators and their continuity

Proofs of our main results will rely on characterization of the continuity of Hankel operators $H_{\overline{m}}$ acting between a space X and $\overline{\mathbf{BMOA}}$. This section presents elementary material on Hankel operators between spaces of analytic functions, and conditions for their continuity. The main results are Proposition 3.1 and Proposition 3.4.

3.1. **Hankel operators with Lipschitz symbols.** We deal first with the case $X = \mathcal{H}^p$ for $p \in (0, \infty)$. Note that the definition of the Hankel operator in (2.4) involving the projection operator P_- in (2.2) presupposes the integrability of $\overline{m}f$ on $\mathbb{T}$. Therefore, in the case that $p < 1$, we may initially only define $H_{\overline{m}}$ on a dense subset of $\mathcal{H}^p$. In any case, let us say that $H_{\overline{m}}$ is continuous as an operator from $\mathcal{H}^p$ into $\overline{\mathbf{BMOA}}$ if there exists a constant $C > 0$ such that for every function $h \in \mathcal{H}^\infty$ we have the estimate

$$\|H_{\overline{m}}h\|_{\overline{\mathbf{BMOA}}} \leq C\|h\|_p, \tag{3.1}$$

where we use the natural definition $\|g\|_{\overline{\mathbf{BMOA}}} := \|\overline{g}\|_{\mathbf{BMOA}}$ and where $\|\cdot\|_p$ was defined in (1.3). If (3.1) holds, then $H_{\overline{m}}$ extends by continuity from the (dense) subset $\mathcal{H}^\infty$ to $\mathcal{H}^p$, and the extension is a continuous linear operator from $\mathcal{H}^p$ into $\overline{\mathbf{BMOA}}$.

Proposition 3.1. *Let $p \in (0, \infty)$ and $m \in \mathcal{H}^\infty$. The Hankel operator $H_{\overline{m}}$ is continuous from $\mathcal{H}^p$ into $\overline{\boldsymbol{BMOA}}$ if and only if $m \in \Lambda^a_{1/p}$.*

The proof is essentially the same as the classical proof of boundedness of the Hankel operator $H_{\overline{m}} : \mathcal{H}^2 \to \overline{z\mathcal{H}^2}$ being equivalent to $m \in \mathbf{BMOA}$. The pivotal point in the classical proof is the ability to factor a function in the predual to $\mathbf{BMOA}$, which is $\mathcal{H}^1$, into a product of two functions in $\mathcal{H}^2$. We follow this idea in our proof of Proposition 3.1.

In order to apply the reasoning in the last paragraph, we need to identify the Cauchy predual space of the analytic Lipschitz classes. This is done in the following result.

Lemma 3.2 (Duren-Romberg-Shields, Hardy-Littlewood)**.** *Let $q \in (0, 1)$ and $m \in \mathcal{H}^\infty$. The following statements are equivalent.*

(i) The limit

$$\ell_m(f) := \lim_{r\to 1-} \int_{\mathbb{T}} f(r\zeta)\overline{m(r\zeta)}|d\zeta|$$

exists for every $f \in \mathcal{H}^q$ and satisfies the bound

$$|\ell_m(f)| \leq C_m\|f\|_q$$

for some constant $C_m > 0$.

(ii) We have $m \in \Lambda^a_{1/q-1}$.

In other words, the Cauchy dual of $\mathcal{H}^q$ is $\Lambda^a_{1/q-1}$. Duren, Romberg and Shields give a proof in [5]. The critical estimate in their proof is an inequality for integral means of analytic functions due to Hardy and Littlewood from [11, page 412].

We need also an elementary factorization result.

Lemma 3.3. *Let the positive real numbers p, q, s be related by*

$$\frac{1}{q} = \frac{1}{s} + \frac{1}{p}.$$

(i) Every function $f \in \mathcal{H}^q$ can be factored as $f = g \cdot h$, with $g \in \mathcal{H}^s$ and $h \in \mathcal{H}^p$ in such a way that

$$\|f\|_q = \|g\|_s \cdot \|h\|_p.$$

(ii) Conversely, given $g \in \mathcal{H}^s$ and $h \in \mathcal{H}^p$, the product $f = gh$ satisfies

$$\|f\|_q \leq \|g\|_s \cdot \|h\|_p$$

Part (i) follows readily from the inner-outer factorization of Hardy space functions, while part (ii) follows from an application of Hölder's inequality.

Proof of Proposition 3.1. Fix $p \in (0, \infty)$ and let $m \in \Lambda^a_{1/p}$. To show that $H_{\overline{m}}$ is continuous from $\mathcal{H}^p$ into $\overline{\mathbf{BMOA}}$ it will suffice to show by the $\mathcal{H}^1 - \mathbf{BMOA}$ duality that we have an estimate of the form

$$\Big| \int_{\mathbb{T}} gH_{\overline{m}}h|d\zeta| \Big| \leq C_m \|g\|_1 \|h\|_p$$

for every pair of functions g and h in $\mathcal{H}^\infty$. Note that the left-hand side in the inequality above vanishes if g is a constant function, and so we may assume that $g(0) = 0$. Set $f = gh$. The expression inside the absolute value on the right-hand side above equals the inner product $\langle P_- \overline{m}h, \overline{g} \rangle$ in $L^2(\mathbb{T})$, and since $\overline{g} \in \overline{z\mathcal{H}^2} = L^2(\mathbb{T}) \ominus \mathcal{H}^2$, it equals

$$\begin{aligned} \int_{\mathbb{T}} gH_{\overline{m}}h|d\zeta| &= \langle \overline{m}h, \overline{g} \rangle \\ &= \int_{\mathbb{T}} \overline{m}gh|d\zeta| \\ &= \int_{\mathbb{T}} \overline{m}f|d\zeta|. \end{aligned}$$

If q satisfies $\frac{1}{q} = 1 + \frac{1}{p}$, then $\Lambda^a_{1/p} = \Lambda^a_{1/q-1}$, and by Lemma 3.2 and part (ii) of Lemma 3.3, last expression is bounded in modulus by

$$C_m \|f\|_q \leq C_m \|g\|_1 \|h\|_p.$$

Thus $H_{\overline{m}}$ is continuous from $\mathcal{H}^p$ into $\overline{\mathbf{BMOA}}$.

If conversely $H_{\overline{m}}$ is continuous from $\mathcal{H}^p$ into $\overline{\mathbf{BMOA}}$, then to show that $m \in \Lambda^a_{1/p}$ it will suffice by Lemma 3.2 to show that

$$\Big| \int_{\mathbb{T}} \overline{f}m|d\zeta| \Big| \leq C\|f\|_q$$

for some constant $C > 0$, where q is as in the proof of the previous implication. If f is a constant function, then the inequality is obvious, and so we may assume that $f(0) = 0$. Factor $f = gh$ according to part (i) of Lemma 3.3, so that $g(0) = 0$, and $\|f\|_q = \|g\|_1 \cdot \|h\|_p$. Then $\overline{g} \in \overline{z\mathcal{H}^2}$, and so

$$\begin{aligned} \int_{\mathbb{T}} \overline{f}m|d\zeta| &= \langle \overline{gh}, \overline{m} \rangle \\ &= \langle \overline{g}, H_{\overline{m}}h \rangle \\ &= \int_{\mathbb{T}} \overline{g}\overline{H_{\overline{m}}h}|d\zeta| \end{aligned}$$

Using the $\mathcal{H}^1 - \mathbf{BMOA}$ duality and the assumption of continuity of $H_{\overline{m}}$, the last expression is bounded in modulus by

$$C\|g\|_1 \cdot \|H_{\overline{m}}h\|_{\overline{\mathbf{BMOA}}} \leq C'\|g\|_1 \cdot \|h\|_p = C'\|f\|_q.$$

The proof is complete. □

3.2. **Hankel operators with Gevrey symbols.** The purpose now is to derive results similar to the ones above, but which apply to Hankel operators with Gevrey symbols defined in (1.2). The domain of the Hankel operator $H_{\overline{m}}$ will be the Privalov class $\mathcal{N}^q$ for $q > 1$, or the Smirnov class $\mathcal{N}^+$. Topologies on these spaces will be induced by the translation-invariant metrics

$$\|f-g\|_{\mathcal{N}^q} := \int_{\mathbb{T}} \big(\log(1+|f-g|)\big)^q |d\zeta|, \quad f,g \in \mathcal{N}^q, \tag{3.2}$$

and

$$\|f-g\|_{\mathcal{N}^+} := \int_{\mathbb{T}} \log(1+|f-g|)|d\zeta|, \quad f,g \in \mathcal{N}^+. \tag{3.3}$$

With these definitions, $\mathcal{N}^q$ and $\mathcal{N}^+$ become so-called F-spaces (see [22, Section 1.8]). A topological vector space X is said to be an F-space if scalar multiplication and addition are continuous in X, and the topology of X is induced by a complete and translation-invariant metric. Whenever we mention the space $\mathcal{N}^q$ or $\mathcal{N}^+$, it is understood that they are equipped with the respective metric topologies.

We say that the Hankel operator $H_{\overline{m}}$ is continuous from $\mathcal{N}^q$ (or $\mathcal{N}^+$) into $\overline{\mathbf{BMOA}}$ if $H_{\overline{m}}$ defined in (2.4) has a continuous extension to an operator from $\mathcal{N}^q$ (or $\mathcal{N}^+$) into $\overline{\mathbf{BMOA}}$.

Proposition 3.4.

(i) For $q > 1$, the Hankel operator $H_{\overline{m}} : \mathcal{N}^q \to \overline{\boldsymbol{BMOA}}$ is continuous if and only if $m \in \mathcal{G}_{1/(1+q)}$.

(ii) The Hankel operator $H_{\overline{m}} : \mathcal{N}^+ \to \overline{\boldsymbol{BMOA}}$ is continuous if and only if $m \in \mathcal{G}_{1/2}$.

In fact, characterization of the continuity of Hankel operators in this case is rather insensitive to change of the range space. In (i), we may replace $\overline{\mathbf{BMOA}}$ by $\overline{\mathcal{N}^q}$ itself, or any of the spaces $\overline{\mathcal{H}^p}$, even for $p = \infty$, without changing the conclusion. So, for instance, membership of m in the corresponding Gevrey class is equivalent to continuity of the operator $H_{\overline{m}} : \mathcal{N}^q \to \overline{\mathcal{H}^p}$. Similar statement holds for the operators $H_{\overline{m}}$ on $\mathcal{N}^+$ also. These facts can be deduced from the proofs below, but they will not be used in the sequel.

To prove Proposition 3.4, we will need counterparts of Lemma 3.2.

Lemma 3.5. *Let $m \in \mathcal{H}^\infty$ and set*

$$\ell_m(f) := \int_{\mathbb{T}} f(\zeta)\overline{m(\zeta)}|d\zeta| = 2\pi \sum_{k=0}^{\infty} \widehat{f}(k)\overline{\widehat{m}(k)}, \quad f \in \mathcal{H}^\infty.$$

(i) For $p > 1$, the mapping ℓ_m extends to a continuous linear functional on $\mathcal{N}^q$ if and only if $m \in \mathcal{G}_{1/(1+q)}$.

(ii) The mapping ℓ_m extends to a continuous linear functional on $\mathcal{N}^+$ if and only if $m \in \mathcal{G}_{1/2}$.

Part (*ii*) is a well known result of Yanagihara from [26], while part (*i*) is due to Meštrović and Pavićević in [17], who gave a proof using earlier results of Stoll from [25]. Moreover, in [25] and [26] the following elementary Taylor coefficient estimates are contained.

Lemma 3.6.

(i) For any $q > 1$ and $f \in \mathcal{N}^q$, we have the estimate

$$|\widehat{f}(n)| = \exp\big(o(n^{1/(1+q)})\big), \quad n \geq 0.$$

(ii) For $f \in \mathcal{N}^+$, we have the estimate

$$|\widehat{f}(n)| = \exp\big(o(n^{1/2})\big), \quad n \geq 0.$$

Proof of Proposition 3.4. Parts (*i*) and (*ii*) have the same proof. For the moment, let us assume that $f \in \mathcal{H}^\infty$ and note that (2.4) implies the Fourier series representation

$$\widehat{H_{\overline{m}}f}(n) = 2\pi \sum_{k=0}^{\infty} \widehat{f}(k)\overline{\widehat{m}(k-n)} = \int_{\mathbb{T}} f(\zeta)\zeta^{|n|}\overline{m(\zeta)}|d\zeta|, \quad n \leq -1 \tag{3.4}$$

and $\widehat{H_{\overline{m}}f}(n) = 0$ for $n \geq 0$. If $H_{\overline{m}}$ is a continuous operator from $\mathcal{N}^q$ (or $\mathcal{N}^+$) to a space on which the mappings $g \mapsto \widehat{g}(n)$ are continuous (in particular, this applies to $\overline{\mathbf{BMOA}}$), then $f \mapsto \widehat{H_{\overline{m}}f}(-1)$ is a continuous linear functional on $\mathcal{N}^q$ (or $\mathcal{N}^+$), and from (3.4) and Lemma 3.5 we deduce readily that m lies in the corresponding Gevrey class.

We will prove the converse statement for case (*ii*) , the proof for the case (*i*) being analogous. Since we are assuming that $m \in \mathcal{G}_{1/2}$, the series $\sum_{k=0}^{\infty} \widehat{f}(k)\overline{\widehat{m}(k-n)}$ converges absolutely for *every* $f \in \mathcal{N}^+$ and $n \leq -1$. In fact, we have

$$\sum_{k=0}^{\infty} \big|\widehat{f}(k)\overline{\widehat{m}(k-n)}\big| \leq A \exp\big(-d|n|^{1/2}\big) \tag{3.5}$$

for some constants $A > 0$ and $d > 0$ depending only on f and m. Accepting for a moment the claim, note that the linear operator

$$f \mapsto 2\pi \sum_{n \leq -1} \Big(\sum_{k=0}^{\infty} \widehat{f}(k)\overline{\widehat{m}(k-n)}\Big)\overline{z^n}$$

maps $\mathcal{N}^+$ into $\overline{\mathcal{G}_{1/2}} \subset \overline{\mathbf{BMOA}}$, coincides for $f \in \mathcal{H}^\infty$ with our earlier definition of $H_{\overline{m}}$ by (3.4), and it is easily seen to be closed (since the linear functionals $f \mapsto \sum_{k=0}^{\infty} \widehat{f}(k)\overline{\widehat{m}(k-n)}$ are continuous on $\mathcal{N}^+$ for each n, by virtue of $m \in \mathcal{G}_{1/2}$ and Lemma 3.5). Since $\mathcal{N}^+$ and $\overline{\mathbf{BMOA}}$ are F-spaces, the usual formulation of the closed graph theorem applies (see [22, Section 2.15]), and we conclude that $H_{\overline{m}} : \mathcal{N}^+ \to \overline{\mathbf{BMOA}}$ is continuous.

It remains to verify the claim (3.5). Since $m \in \mathcal{G}_{1/2}$, we have that $|\widehat{m}(k)| \leq B\exp(-3dk^{1/2})$ for some $B > 0$, $d > 0$ and every integer $k \geq 0$. By part (*ii*) of Lemma 3.6, there exists a positive integer K such that $|\widehat{f}(k)| \leq \exp(dk^{1/2})$ for $k \geq K$. Then, for integers $n \leq -1$, we have

$$\begin{aligned}\big|\widehat{f}(k)\overline{\widehat{m}(k-n)}\big| &\leq B\exp(-2d(k+|n|)^{1/2}) \\ &\leq B\exp(-d|n|^{1/2})\exp(-dk^{1/2})\end{aligned}$$

for $k \geq K$. If $D = \max\{|\widehat{f}(k)| : 0 \leq k < K\}$, we may now estimate

$$\begin{aligned}\sum_{k=0}^{\infty} \left|\widehat{f}(k)\overline{\widehat{m}(k-n)}\right| &\leq DK \exp(-d|n|^{1/2}) + B \exp(-d|n|^{1/2}) \sum_{k=K}^{\infty} \exp(-dk^{1/2}) \\ &\leq A \exp(-d|n|^{1/2}).\end{aligned}$$

We have verified (3.5), and so the proof is complete (in the proof of case (i), the above estimates are all valid with the exponent $1/2$ replaced by $1/(1+q)$). □

4. Pythagorean factorizations

To each symbol b we have associated the quotient $h = b/a \in \mathcal{N}^+$. This process can be reversed, and so every $h \in \mathcal{N}^+$ can be uniquely expressed as a quotient of Pythagorean mates. In this section, we prove a stability result for this factorization.

4.1. Pythagorean factorization and their stability.

Let $h \in \mathcal{N}^+$ and let I be the inner factor of h. Set a to be the unique outer function satisfying $a(0) > 0$ and

$$|a|^2 = \frac{1}{|h|^2 + 1} \tag{4.1}$$

almost everywhere on $\mathbb{T}$. Then there exists a unimodular scalar c and a unique outer function b_o which satisfies $b_o(0) > 0$,

$$|b_o|^2 = \frac{|h|^2}{|h|^2 + 1}$$

and

$$h(z) = \frac{cI(z)b_o(z)}{a(z)} = \frac{b(z)}{a(z)}, \quad z \in \mathbb{D}.$$

We say that $h = b/a$ is the *Pythagorean factorization* of h. Clearly b and a are Pythagorean mates, in the sense that the equality $|b|^2 + |a|^2 = 1$ holds almost everywhere on $\mathbb{T}$.

We will need a convergence result for mates in the factorizations.

Proposition 4.1. *Assume that the sequence of functions $\{h_n\}_n$ converges to the function h in the metric topology on $\mathcal{N}^+$. If $h_n = b_n/a_n$ and $h = b/a$ are the corresponding Pythagorean factorizations, then there exists a subsequence $\{n_k\}_k$ such that*

$$\lim_{k \to \infty} a_{n_k}(\zeta) = a(\zeta)$$

and

$$\lim_{k \to \infty} b_{n_k}(\zeta) = b(\zeta)$$

for almost every $\zeta \in \mathbb{T}$.

We prove Proposition 4.1 after a brief review of basic properties of outer functions.

4.2. **Pointwise convergence of outer functions on the boundary.** Recall that if a is an outer function satisfying $a(0) > 0$, and we set $g(\zeta) = \log|a(\zeta)|$, then we have the representation formula

$$\log a(z) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} g(\zeta)|d\zeta| = \widehat{g}(0) + 2\sum_{n\geq 1}\widehat{g}(n)z^n = 2P_+g(z) - \widehat{g}(0), \quad z \in \mathbb{D}.$$

For any $p \in (0,1)$ the operator P_+ is continuous from $L^1(\mathbb{T})$ into $\mathcal{H}^p$ (see, for instance, [1, Theorem 2.1.10]). Thus if $g_n(\zeta) = \log|a_n(\zeta)|$ and $g_n \to g$ in $L^1(\mathbb{T})$, then after passing to a subsequence we can ensure the pointwise convergence

$$\lim_n \log a_n(\zeta) = \log a(\zeta)$$

for almost every $\zeta \in \mathbb{T}$. After exponentiating this translates into

$$\lim_n a_n(\zeta) = a(\zeta)$$

for almost every $\zeta \in \mathbb{T}$. The proof of Proposition 4.1 is thus reduced to showing that the convergence $h_n = b_n/a_n \to h = b/a$ in $\mathcal{N}^+$ implies that $\log|a_{n_k}| \to \log|a|$ in $L^1(\mathbb{T})$ along some subsequence $\{n_k\}_k$. For if this is the case, then since $h_n \to h$ in $\mathcal{N}^+$ implies by (3.3) that

$$\lim_{n\to\infty}\int_{\mathbb{T}} \log(1+|h_n - h|)|d\zeta| = 0, \tag{4.2}$$

basic measure theory ensures that the subsequence can be refined to ensure almost everywhere pointwise convergence $h_{n_k}(\zeta) \to h(\zeta)$. By the above reasoning we can ensure also the convergence $a_{n_k}(\zeta) \to a(\zeta)$ almost everywhere, and then it follows that $b_{n_k}(\zeta) \to b(\zeta)$ almost everywhere.

4.3. **Proof of the stability result.** The sought-after result now follows in a standard way from uniform integrability (see [21, page 133], for instance). Recall that a sequence $\{g_n\}_n$ of functions in $L^1(\mathbb{T})$ is said to be uniformly integrable if

$$\lim_{\delta\to 0}\sup_{E:|E|\leq\delta}\int_E |g_n||d\zeta| = 0$$

holds uniformly in n. An elementary argument shows that if $g_n \to g$ almost everywhere on $\mathbb{T}$, then uniform integrability of the sequence $\{g_n\}_n$ is equivalent to convergence $g_n \to g$ in $L^1(\mathbb{T})$.

Proof of Proposition 4.1. According to the above discussion, it suffices for us to show that $\log|a_n| \to \log|a|$ in $L^1(\mathbb{T})$. Since $h_n \to h$ in $\mathcal{N}^+$, after passing to a subsequence, we may suppose that $h_n(\zeta) \to h(\zeta)$ almost everywhere on $\mathbb{T}$. Recalling (4.1), this implies that

$$\lim_n |a_n(\zeta)|^2 = \lim_n \frac{1}{|h_n(\zeta)|^2+1} = \frac{1}{|h(\zeta)|^2+1} = |a(\zeta)|^2$$

almost everywhere on $\mathbb{T}$. Taking logarithms, we obtain $\log|a_n(\zeta)| \to \log|a(\zeta)|$ almost everywhere on $\mathbb{T}$.

Note that $1/|a|^2 = |h_n|^2 + 1$ on $\mathbb{T}$ and recall that $\sqrt{x+1} \leq \sqrt{x}+1$ for $x \geq 0$. Use also that the logarithm is increasing to see that on $\mathbb{T}$ we have

$$\begin{aligned}\big|\log|a_n|\big| &= \log\big(\sqrt{|h_n|^2+1}\big)\\ &\leq \log\big(|h_n|+1\big)\\ &\leq \log\big(1+|h_n-h|\big)+\log\big(1+|h|\big).\end{aligned}$$

Since (4.2) holds, the sequence $\{\log(1+|h_n-h|)\}_n$ converges in $L^1(\mathbb{T})$ (to 0), and so is uniformly integrable. But then the above inequalities show that $\{\log|a_n|\}_n$ is uniformly integrable on $\mathbb{T}$, and since $\log|a_n| \to \log|a|$ almost everywhere on $\mathbb{T}$, we have also $\log|a_n| \to \log|a|$ in $L^1(\mathbb{T})$. By the initial remarks, the proof is complete. □

5. Proof of the main theorems

5.1. **A general criterion.** Theorem A and Theorem B will follow from our earlier developments as corollaries of a general statement which we shall now state, and prove next.

In this section, we let X be a topological space of analytic functions on $\mathbb{D}$. The properties of X which we shall need in our proof are as follows.

(i) X is an F-space.
(ii) X is continuously contained in the Smirnov class $\mathcal{N}^+$.
(iii) $\mathcal{H}^\infty$ is dense in X.
(iv) The multiplication operator $f \mapsto \varphi f$ is continuous on X for each $\varphi \in \mathcal{H}^\infty$.
(v) If $h \in X$ has Pythagorean factorization $h = b/a$, then $1/a \in X$.

It is known that the spaces appearing in theorems stated in the Introduction, namely $\mathcal{H}^p$, $\mathcal{N}^q$ and $\mathcal{N}^+$, all satisfy the above five conditions. See, for instance, [10], [20] and [25]. The fifth property is easily seen by recalling that $1/|a|^2 = |h|^2+1$ on $\mathbb{T}$.

Our general result will apply to any space X satisfying the above five conditions. To such X we associate two sets of functions. As before, we let $\mathcal{F}(X)$ consist of non-extreme symbols b with Pythagorean mate a for which b/a is a member of X, and we let $H(X, \overline{\mathbf{BMOA}})$ consist of those analytic functions $m \in \mathcal{H}^\infty$ for which the densely defined Hankel operator $H_{\overline{m}}$ in (2.4) extends to a continuous mapping from X into $\overline{\mathbf{BMOA}}$.

Proposition 5.1. *Let X be a topological space of analytic functions on $\mathbb{D}$ satisfying properties $(i)-(v)$ stated above. Then the universal multipliers for $\mathcal{F}(X)$ coincide with the symbols of continuous Hankel operators $H_{\overline{m}} : X \to \overline{\boldsymbol{BMOA}}$. That is, we have the equality*

$$\bigcap_{b\in\mathcal{F}(X)} \boldsymbol{Mult}(\mathcal{H}(b)) = H(X, \overline{\boldsymbol{BMOA}}).$$

Theorem A, Theorem B and the Davis-McCarthy Theorem follow immediately from Proposition 5.1 and the characterization of continuous Hankel operators in Proposition 3.1 and Proposition 3.4.

We proceed with the proof of Proposition 5.1.

5.2. **Toeplitz operators.** For notational convenience, we shall use the Toeplitz operators

$$\mathcal{T}_g : \mathcal{H}^2 \to \mathcal{H}^2, \quad f \mapsto P_+(\overline{g}f). \tag{5.1}$$

Here $g \in L^\infty(\mathbb{T})$ is the symbol of the operator. The following commutation relation is readily established and will be used frequently:

$$\mathcal{T}_{\overline{g_1 g_2}} = \mathcal{T}_{\overline{g_1}}\mathcal{T}_{\overline{g_2}} = \mathcal{T}_{\overline{g_2}}\mathcal{T}_{\overline{g_1}}, \quad g_1, g_2 \in \mathcal{H}^\infty. \tag{5.2}$$

If $g \in \mathcal{H}^\infty$, then the operator $T_{\overline{g}}$ is bounded from **BMOA** into itself. To see this, we may simply observe that $\mathcal{T}_{\overline{g}}$ is the Banach space adjoint of the operator of multiplication by g on $\mathcal{H}^1$, with respect to the $\mathcal{H}^1 - \mathbf{BMOA}$ duality pairing in (2.5). Moreover, if g is an outer function, then the operator $\mathcal{T}_{\overline{g}}$ is injective on $\mathcal{H}^2$.

5.3. **From Hankel continuity to universal multiplier.** We will first prove that if $m \in H(X, \overline{\mathbf{BMOA}})$, then m is a universal multiplier for the family $\mathcal{F}(X)$. Let $b \in \mathcal{F}(X)$, so that $b/a \in X$. By properties (iii) and (v) in Section 5.1, there exists a sequence $\{h_n\}_n$ of functions in $\mathcal{H}^\infty$ such that $h_n \to 1/a$ in the topology of X. By property (iv), we have $ah_n \to 1$ in X. Now, $H_{\overline{m}}(a^{-1}) \in \overline{\mathbf{BMOA}}$, and this function vanishes at $z = 0$. Say,

$$H_{\overline{m}}(a^{-1}) = P_-(\overline{m}a^{-1}) = \overline{zu},$$

where $u \in \mathbf{BMOA}$. Using the identity (2.3) and continuity of $H_{\overline{m}}$ we obtain

$$zu = \lim_n \overline{H_{\overline{m}}h_n} = \lim_n P_+(\overline{h_n}m) + c_n = \lim_n \mathcal{T}_{\overline{h_m}}m + c_n, \tag{5.3}$$

with convergence in the sense of the norm on **BMOA** and where c_n are constants. Applying the operator $\mathcal{T}_{\overline{z}}$ (here z denotes the identity function on $\mathbb{T}$) and using the relation (5.2), we arrive at

$$u = \lim_n \mathcal{T}_{\overline{zh_n}}m.$$

Applying also the Toeplitz operator $\mathcal{T}_{\overline{a}}$, which, as mentioned above, is continuous on **BMOA**, and using (2.3), we obtain

$$\begin{aligned}
\mathcal{T}_{\overline{a}}u &= \lim_n \mathcal{T}_{\overline{zah_n}}m \\
&= \lim_n P_+(\overline{azh_n}m) \\
&= \lim_n \overline{H_{\overline{m}}(azh_n)} + \lim_n c'_n
\end{aligned}$$

where c'_n are constants. Since $ah_n \to 1$ in X, the continuity of the Hankel operator $H_{\overline{m}}$ implies that

$$H_{\overline{m}}(azh_n) \to H_{\overline{m}}z = \overline{\frac{m - m(0)}{z}} - \overline{m'(0)}$$

in $\overline{\mathbf{BMOA}}$. We must then also have that $\lim_n c'_n = c'$ for some constant c'. Finally, we obtain that

$$\mathcal{T}_{\overline{a}}u = \frac{m - m(0)}{z} - m'(0) + c'.$$

Since $u \in \mathbf{BMOA}$, we obtain by Corollary 2.2 that $\frac{m-m(0)}{z} - m'(0) + c' \in \mathbf{Mult}(\mathcal{H}(b))$. It follows that $m \in \mathbf{Mult}(\mathcal{H}(b))$.

We have therefore proved that the right-hand side is included in the left-hand side in the asserted equality in the statement of Proposition 5.1.

5.4. **From universal multiplier to Hankel continuity.** It takes considerably more effort to prove that the universal multiplier property of m for the family $\mathcal{F}(X)$ implies that $m \in H(X, \overline{\mathbf{BMOA}})$. We start with a few observations.

If m is a universal multiplier for the family $\mathcal{F}(X)$, then for any $h = b/a \in X$ in particular we have $m \in \mathcal{H}(b)$, and so a unique mate $m_+(h) \in \mathcal{H}^2$ exists which satisfies the operator equation

$$\mathcal{T}_{\overline{b}} m = \mathcal{T}_{\overline{a}} m_+(h). \tag{5.4}$$

The use of notation $m_+(h)$ in favor of the more natural $m_+(b)$ is a conscious choice, as we will soon see that $h \mapsto \overline{m_+(h)}$ is a linear function. Let us note that our assumptions force $m_+(h)$ to lie in **BMOA**. By property (iv) in Section 5.1, if $b/a \in X$ and I is an inner function, then $Ib/a \in X$. But then $Ib \in \mathcal{F}(X)$, since a is the Pythagorean mate of Ib. Hence $\mathcal{F}(X)$ satisfies the hypothesis of Corollary 2.4. Thus the mapping

$$\overline{m_+} : X \to \overline{\mathbf{BMOA}}, \quad h = b/a \mapsto \overline{m_+(h)} \tag{5.5}$$

is well-defined. Our task will be to show that $\overline{m_+}$ is linear and continuous. At the end of our development, we will note that $\overline{m_+}$ is a rank one perturbation of the operator $H_{\overline{m}} : X \to \overline{\mathbf{BMOA}}$.

We start by establishing linearity.

Lemma 5.2. *For $h_1, h_2 \in X$ and a scalar $\lambda \in \mathbb{C}$, we have*

$$\overline{m_+(\lambda h_1 + h_2)} = \lambda \overline{m_+(h_1)} + \overline{m_+(h_2)}.$$

Proof. Let $h = \lambda h_1 + h_2$ and consider the Pythagorean factorizations

$$h_1 = \frac{b_1}{a_1}, \quad h_2 = \frac{b_2}{a_2}, \quad h = \frac{b}{a}.$$

We have the identity

$$a_1 a_2 b = \lambda a a_2 b_1 + a a_1 b_2,$$

and so

$$\mathcal{T}_{\overline{a_1 a_2 b}} m = \overline{\lambda} \mathcal{T}_{\overline{a a_2 b_1}} m + \mathcal{T}_{\overline{a a_1 b_2}} m. \tag{5.6}$$

The relation (5.2) gives

$$\mathcal{T}_{\overline{a_1 a_2 b}} m = \mathcal{T}_{\overline{a_1 a_2}} \mathcal{T}_{\overline{b}} m = \mathcal{T}_{\overline{a_1 a_2}} \mathcal{T}_{\overline{a}} m_+(h) = \mathcal{T}_{\overline{a a_1 a_2}} m_+(h).$$

Similarly

$$\mathcal{T}_{\overline{a a_2 b_1}} m = \mathcal{T}_{\overline{a a_1 a_2}} m_+(h_1)$$

and

$$\mathcal{T}_{\overline{a a_1 b_2}} m = \mathcal{T}_{\overline{a a_1 a_2}} m_+(h_2).$$

Inputting these equalities into the relation (5.6) and rearranging, we obtain

$$\mathcal{T}_{\overline{a a_1 a_2}} \big(m_+(h) - \overline{\lambda} m_+(h_1) - m_+(h_2) \big) = 0.$$

Because $a a_1 a_2$ is an outer function, the above equality implies that $m_+(h) - \overline{\lambda} m_+(h_1) - m_+(h_2) = 0$, and so the proof is complete. □

We treat continuity next.

Lemma 5.3. *The linear operator $\overline{m_+} : X \to \overline{\boldsymbol{BMOA}}$ is continuous.*

Proof. Since X is an F-space by property (i) in Section 5.1, the usual formulation of the closed graph theorem is applicable to our situation (see [22, Section 2.15]). Hence the operator $\overline{m_+}$ will be continuous if we can verify the validity of the implication

$$\begin{cases} h_n \to h & \text{in } X \\ \overline{m_+(h_n)} \to \overline{g} & \text{in } \overline{\mathbf{BMOA}} \end{cases} \quad \Rightarrow \quad \overline{m_+(h)} = \overline{g}.$$

Above we assume that $g \in \mathbf{BMOA}$, so that $m_+(h_n) \to g$ in the norm of $\mathbf{BMOA}$ (recall that we have set $\|g\|_{\mathbf{BMOA}} := \|\overline{g}\|_{\overline{\mathbf{BMOA}}}$). Let the two above hypotheses be satisfied. Since X is continuously contained in $\mathcal{N}^+$ by property (ii) in Section 5.1, by Proposition 4.1 and by passing to a subsequence we may assume that we have the pointwise convergence $b_n \to b$ and $a_n \to a$ almost everywhere on $\mathbb{T}$, where $h_n = b_n/a_n$ and $h = b/a$ are the corresponding Pythagorean factorizations. Since the mapping $f \mapsto \mathcal{T}_{\overline{a}} f$ is continuous on $\mathbf{BMOA}$, we have

$$\begin{aligned} \mathcal{T}_{\overline{a}} g &= \lim_n \mathcal{T}_{\overline{a}} m_+(h_n) \\ &= \lim_n \mathcal{T}_{\overline{a - a_n}} m_+(h_n) + \mathcal{T}_{\overline{a_n}} m_+(h_n) \\ &= \lim_n \mathcal{T}_{\overline{a - a_n}} m_+(h_n) + \mathcal{T}_{\overline{b_n}} m \end{aligned} \tag{5.7}$$

The convergence above holds in the norm of $\mathbf{BMOA}$, and so in particular in the sense of pointwise convergence on $\mathbb{D}$. Now,

$$\mathcal{T}_{\overline{b_n}} m \to \mathcal{T}_{\overline{b}} m \tag{5.8}$$

in (say) $\mathcal{H}^2$, and so pointwise on $\mathbb{D}$, since $b_n \to b$ almost everywhere on $\mathbb{T}$. Since $m_+(h_n) \to g$ in $\mathbf{BMOA}$, the same is true in the space $\mathcal{H}^4$. Therefore, by contractivity of the projection P_+ on $L^2(\mathbb{T})$ and Hölder's inequality, we may estimate

$$\begin{aligned} \|\mathcal{T}_{\overline{a - a_n}} m_+(h_n)\|_2^2 &= \|P_+[\overline{(a - a_n)} m_+(h_n)]\|_2^2 \\ &\leq \|\overline{(a_n - a)} m_+(h_n)\|_2^2 \\ &\leq \|a_n - a\|_4^2 \cdot \|m_+(h_n)\|_4^2. \end{aligned}$$

Since $a_n \to a$ almost everywhere on $\mathbb{T}$, we obtain $\|\mathcal{T}_{\overline{a - a_n}} m_+(h_n)\|_2 \to 0$, and from (5.7) and (5.8) we conclude that

$$\mathcal{T}_{\overline{a}} g(z) = \mathcal{T}_{\overline{b}} m(z), \quad z \in \mathbb{D}.$$

According to (5.4), this means that $g = m_+(h)$. We have thus verified that $\overline{m_+} : X \to \overline{\mathbf{BMOA}}$ is a closed operator. By the closed graph theorem this operator is also continuous. □

Completion of the proof of Proposition 5.1. Recalling the result of Section 5.3, what remains to be established is that any universal multiplier m for $\mathcal{F}(X)$ induces a continuous Hankel operator $H_{\overline{m}} : X \to \overline{\mathbf{BMOA}}$. By Lemma 5.3, for every $h = b/a \in X$, the equation (5.4) has a unique solution $m_+(h)$ which depends continuously on $h \in X$. Let us suppose that $h = b/a \in \mathcal{H}^\infty$. Then $1/a \in \mathcal{H}^\infty$, and we may apply the Toeplitz operator $\mathcal{T}_{1/\overline{a}}$ and relation (5.2) to the mate equation in (5.4) to obtain

$$\mathcal{T}_{\overline{h}} m = m_+(h).$$

Since $\mathcal{T}_{\overline{h}} m = P_+ \overline{h} m$, the relation (2.3) implies that

$$\overline{m_+(h)} = \overline{\mathcal{T}_{\overline{h}} m} = H_{\overline{m}} h + c(h) \tag{5.9}$$

for a constant $c(h)$ which satisfies

$$|c(h)| = |\widehat{\overline{m}h}(0)|.$$

In other words, $c(h)$ is the value at $z = 0$ of the function $m_+(h)$. Since evaluations are continuous on **BMOA**, it follows that $h \mapsto m_+(h)(0)$ is a continuous linear functional defined on all of X. For $h \in \mathcal{H}^\infty$, we have from (5.9) the equality

$$H_{\overline{m}} h = \overline{m_+(h)} - m_+(h)(0).$$

Since in the right-hand side of this equality appears a continuous linear mapping $X \to \overline{\mathbf{BMOA}}$, the left-hand side extends to a continuous linear operator $X \to \overline{\mathbf{BMOA}}$. That is, $m \in H(X, \overline{\mathbf{BMOA}})$. $\square$

6. Further comments and remarks

We end the article with a list of a few unresolved matters and ideas for further research.

6.1. **Range spaces of coanalytic Toeplitz operators.** One consequence of following the presented new approach to universal multipliers is that we never needed to characterize the corresponding intersection of co-analytic Toeplitz operator ranges, which is a necessary step in the Davis-McCarthy approach. Denoting by $M(\overline{\varphi})$ the range of the co-analytic Toeplitz operator $\mathcal{T}_{\overline{\varphi}} : \mathcal{H}^2 \to \mathcal{H}^2$, a consequence of results from [3] is that we have

$$\mathcal{G}_{1/2} = \bigcap_{\varphi \in \mathcal{H}^\infty \setminus \{0\}} M(\overline{\varphi}) = \bigcap_{\varphi \in \mathcal{H}^\infty \setminus \{0\}} \mathbf{Mult}(M(\overline{\varphi})) = \bigcap_{b \in \mathcal{F}(\mathcal{N}^+)} \mathbf{Mult}(\mathcal{H}(b)).$$

Naively, one may expect from our main result that we should have a similar equality, with $\mathcal{G}$ replaced by $\Lambda^a_{1/p}$, and the intersections corresponding to spaces $M(\overline{\varphi})$ restricted to those (outer, bounded) symbols φ for which $1/\varphi \in \mathcal{H}^p$, namely

$$\Lambda^a_{1/p} = \bigcap_{\varphi : 1/\varphi \in \mathcal{H}^p} M(\overline{\varphi}) = \bigcap_{\varphi : 1/\varphi \in \mathcal{H}^p} \mathbf{Mult}(M(\overline{\varphi})).$$

That this is not the case in general is seen by setting $p = 2$. Then, for every $m \in \mathcal{H}^\infty$ and $1/\varphi \in \mathcal{H}^2$, we have $m = \mathcal{T}_{\overline{\varphi}} g$ where $g = P_+(m/\overline{\varphi})$, so $\mathcal{H}^\infty \subseteq \bigcap_{1/\varphi \in \mathcal{H}^2} M(\overline{\varphi})$. However, the equality

$$\Lambda^a_{1/p} = \bigcap_{\varphi : 1/\varphi \in \mathcal{H}^p} \mathbf{Mult}(M(\overline{\varphi}))$$

is not ruled out by this argument. We conjecture that this equality holds.

6.2. **Other families of symbols.** In this article, we focused on what we considered natural classes of symbols $\mathcal{F}(\mathcal{H}^p)$ and $\mathcal{F}(\mathcal{N}^q)$, but similar questions may of course be explored about universal multipliers for families $\mathcal{F}(X)$ corresponding to any other reasonable space X. In relation to this, recall that we have characterized the multipliers of symbol families $\mathcal{F}(\mathcal{H}^p)$ for finite p, and we note also that the case $b/a \in \mathcal{H}^\infty$ corresponds to the equality $\mathcal{H}(b) = \mathcal{H}^2$, with equivalence of norms. So $\mathbf{Mult}(\mathcal{H}(b)) = \mathcal{H}^\infty$ if $b/a \in \mathcal{H}^\infty$. We have not been able to compute the universal multipliers for the intermediate case $b \in \mathcal{F}(\mathbf{BMOA})$. Our Proposition 5.1 does not apply in this case, since **BMOA** does not satisfy the hypotheses of that proposition. What are the universal multipliers in this case?

6.3. **Relating containment of a space within the multiplier algebra.** For $p > 2$, the authors own previous article [14] allows for a different way to express our main theorem: there we showed that $b/a \in \mathcal{H}^{\frac{2p}{p-2}}$ is equivalent with $\mathcal{H}^p \subset \mathcal{H}(b)$ (see Theorem A in [14]). This makes plausible a description of universal multipliers for classes $\mathcal{F}$ described in terms of containing other fixed spaces of analytic functions. The case of the Dirichlet space promises to offer some resistance.

Acknowledgements. The authors were funded through grant PID2023-149061NA-I00 by the Generación de Conocimiento programme and grant RYC2021-034744-I by the Ramón y Cajal programme from Agencia Estatal de Investigación (Spanish Ministry of Science, Innovation and Universities).

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