SHIFT OPERATORS, CAUCHY INTEGRALS AND APPROXIMATIONS

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Abstract. This article consists of two connected parts, results of which might be of independent interest. In the first part, we study the shift invariant subspaces in certain $\mathcal{P}^2(\mu)$-spaces, which are the closures of analytic polynomials in the Lebesgue spaces $\mathcal{L}^2(\mu)$ defined by a class of measures μ living on the closed unit disk $\overline{\mathbb{D}}$. The measures μ which occur in our study have a part on the open disk $\mathbb{D}$ which is radial and decreases at least exponentially fast near the boundary. Our focus is on those shift invariant subspaces which are generated by a bounded function in H^∞. In this context, our results are definitive. We give a characterization of the cyclic singular inner functions by an explicit and readily verifiable condition, and we establish certain permanence properties of non-cyclic ones which are important in the applications. Our applications take up the second part of the article, and concerns removal of singularities, à la Khrushchev, of Cauchy integrals, a problem which is in many ways similar to the problem of characterizing density of certain classes of functions in de Branges-Rovnyak spaces $\mathcal{H}(b)$. We prove that if a function $g \in L^1(\mathbb{T})$ on the unit circle $\mathbb{T}$ has a Cauchy transform with Taylor coefficients of order $O(\exp(-c\sqrt{n})$ for some $c > 0$, then the set $U = \{x \in \mathbb{T} : |g(x)| > 0\}$ is essentially open and $\log|g|$ is locally integrable on U. Another one of our results establishes a simple characterization of those symbols b with the property that the space $\mathcal{H}(b)$ contains a dense subset of functions which, in a sense, just barely fail to have an analytic continuation to a disk of radius larger than 1. We indicate how close our results are to being optimal and pose a few questions.

1. Introduction and main results

1.1. **Some background.** In the first part of the article we will study spaces of analytic functions defined by the condition of their square-integrability against a Borel measure of the form

$$d\mu(z) = G(1-|z|)\,dA(z) + w(z)dm(z), \tag{1}$$

where dA and dm are the area and arc-length measures on, respectively, the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and its boundary circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The radial weight $G(1-|z|)$ living on $\mathbb{D}$ is defined in terms of some continuous, increasing and positive function G, and the weight w living on $\mathbb{T}$ is a general Borel-measurable non-negative integrable function. Given such a measure, we may construct first the classical Lebesgue space $\mathcal{L}^2(\mu)$ of (equivalence classes of) Borel-measurable functions living on the carrier of μ, and next consider its subspace $\mathcal{P}^2(\mu)$, by which we denote the smallest closed subspace of $\mathcal{L}^2(\mu)$ which contains the set $\mathcal{P}$ of analytic polynomials. The space $\mathcal{P}^2(\mu)$ will be the setting for the first part of our study.

The *shift* operator $M_z : \mathcal{P}^2(\mu) \to \mathcal{P}^2(\mu)$, which takes a function $f(z)$ to $zf(z)$, is a *subnormal* operator, in the sense that it is the restriction of a normal operator, namely $M_z : \mathcal{L}^2(\mu) \to \mathcal{L}^2(\mu)$, to an invariant subspace. From the point of view of an operator theorist, the significance of the pair $(\mathcal{P}^2(\mu), M_z)$ lies in the fact that a straight-forward application of the spectral theorem for normal operators essentially reduces the study of the class of subnormal operators to the study of the operator $M_z : \mathcal{P}^2(\mu) \to \mathcal{P}^2(\mu)$. The monograph [6] by Conway is an excellent source of information on subnormal operators.

For measures such as (1), the space $\mathcal{P}^2(\mu)$ is, like $\mathcal{L}^2(\mu)$, a space of Borel-measurable functions on the closed disk $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$. In certain cases it is even a space of *analytic* functions on $\mathbb{D}$. In such a case, each element $f \in \mathcal{P}^2(\mu)$, a priori interpreted as a Borel-measurable function on $\overline{\mathbb{D}}$, has a *unique* restriction $f_{\mathbb{D}}$ to the disk $\mathbb{D}$. The restriction $f_{\mathbb{D}}$ must be an analytic function by the virtue of it being a locally uniform limit of analytic polynomials. We will below use the term *irreducible* for such a space which is in this sense "analytic". It is a difficult problem (and in general open) to determine which weight pairs (G, w) as in (1) produce an irreducible space. Khrushchev in his PhD thesis and in the article [14] solved certain special cases of the problem. For instance, his results apply to $G(t) = t^n$ for some $n > 0$, and $w = 1_E$ being a characteristic function of certain classes of sets $E \subset \mathbb{T}$ (namely, defined in terms of Beurling-Carleson conditions, but we shall not need to discuss these concepts in the present article). Already these results have fascinating applications to function theory, of which there are plenty in [14]. The article [24] builds on Khrushchev's work, explains the structure of $\mathcal{P}^2(\mu)$ when $w = 1_E$ and E is a general subset of $\mathbb{T}$, and showcases further applications to the theory of Cauchy integral operator, uncertainty-type principles, and de Branges-Rovnyak spaces.

1.2. **Irreducible $\mathcal{P}^2(\mu)$-spaces.** Recently, the author found in [23] an exact condition for irreducibility of $\mathcal{P}^2(\mu)$ in the case when $G(t)$ decays at least exponentially as $t \to 0^+$, thus confirming a conjecture by Kriete and MacCluer from [18]. Roughly speaking, if $G(t)$ is smaller than the weight $\exp(-ct^{-1})$ for some $c > 0$, or more precisely if

$$\liminf_{x \to 0^+} x \log 1/G(x) > 0, \qquad \text{(ExpDec)}$$

but large enough to satisfy

$$\text{(LogLogInt)} \qquad \int_0^d \log\log(1/G(x))\,dx < \infty.$$

then the space $\mathcal{P}^2(\mu)$ is irreducible if and only if the carrier set of the measure $d\mu_{\mathbb{T}} = w\,dm$ on $\mathbb{T}$ can be covered by intervals I satisfying the condition

$$\int_I \log w\,dm > -\infty. \tag{2}$$

In order to properly state the result we will need to define the following concept of core sets. For our purposes this concept is critical, and it will appear frequently throughout the article.

Definition 1.1. (Core sets of weights) Let w be a non-negative Lebesgue-integrable function on $\mathbb{T}$. We define $\text{core}(w)$ to be the union of all open intervals I for which (2) holds. In other words,

$$\text{core}(w) = \{x \in \mathbb{T} : \text{ there exists } I \text{ containing } x \text{ for which (2) holds } \} \tag{3}$$

The set $\text{core}(w)$ is open, and it does not depend on the particular representative of w in the space $\mathcal{L}^1(\mathbb{T})$ of equivalence classes of functions which are Lebesgue-integrable on $\mathbb{T}$ with respect to the arc-length measure dm. By elementary measure theory, we may thus assume that w is Borel-measurable.

Definition 1.2. (Carrier sets) Let η be a non-negative Borel measure on $\mathbb{T}$. A Borel subset E of $\mathbb{T}$ is a *carrier* for η if

$$\eta(\mathbb{T} \setminus E) = 0.$$

If w is a Borel-measurable function on $\mathbb{T}$, then we say that a set E is a carrier for w if it is a carrier for the Borel measure $w\,dm$.

Carriers are obviously not unique. The set

$$\{x \in \mathbb{T} : w(x) > 0\} \tag{4}$$

is a carrier for w. If w is only defined up to a set of m-measure zero, then we may take as a carrier for w any set differing from (4) by a set of m-measure zero. Since $\log 0 = -\infty$, it is obvious from (2) that $\text{core}(w)$ is essentially contained in any carrier of w.

The main theorem in [23] shows that the irreducibility of $\mathcal{P}^2(\mu)$-spaces of the form (1) with G satisfying (ExpDec) and (LogLogInt) can be characterized in terms core sets. The following definition is thus justified by [23, Theorem A], with the non-trivial part being the equivalence of the condition (*iii*) with the other two.

Definition 1.3. (Irreducible spaces) A space $\mathcal{P}^2(\mu)$ defined by a measure μ of the form (1), with G satisfying (ExpDec) and (LogLogInt), will be called *irreducible* if any one (and so all) of the following equivalent conditions hold:

(i) the space $\mathcal{P}^2(\mu)$ contains no non-trivial characteristic functions of measurable subsets of $\overline{\mathbb{D}}$: if A is a Borel subset of $\overline{\mathbb{D}}$ and $1_A \in \mathcal{P}^2(\mu)$ is not the zero element, then $1_A = 1_{\overline{\mathbb{D}}}$.
(ii) the space $\mathcal{P}^2(\mu)$ is a space of analytic functions on $\mathbb{D}$ in which the analytic polynomials are dense,
(iii) the set $\text{core}(w)$ is a carrier for w, or in other words it coincides with (4), up to a set of m-measure zero.

If the $\text{core}(w)$ is not a carrier of w, then the space $\mathcal{P}^2(\mu)$ will contain a full Lebesgue space $\mathcal{L}^2(w_r dm)$. Here w_r denotes a certain *residual* weight. The residuals play no role in the statements of our main results, but will be important in the proofs. Their definition is postponed to coming sections.

In particular, the result applies to measures μ of the form

$$(T1) \qquad d\mu(z) = \exp\Big(-\frac{c}{(1-|z|)^\beta}\Big)dA(z) + w(z)dm(z), \quad c > 0,\, \beta \geq 1$$

and

$$(T2) \qquad d\mu(z) = \exp\left(-c\exp\left(\frac{1}{(1-|z|)^\alpha}\right)\right)dA(z) + w(z)dm(z), \quad c > 0,\, \alpha \in (0,1).$$

The present article in essence concerns itself with the theory and various applications of the spaces appearing in Definition 1.3. For instance, spaces defined by measures of the form $(T1)$ will appear in the second part of our paper, when we deal with applications of our developed $\mathcal{P}^2(\mu)$-theory to model spaces K_θ and de Branges-Rovnyak spaces $\mathcal{H}(b)$. The measures $(T2)$ are good to keep in mind as simple examples in which various computations can be readily carried out and to which our theory applies.

The reader might wonder what happens in the case $\beta < 1$ in $(T1)$. In this case, the space $\mathcal{P}^2(\mu)$ is still a space of analytic functions, but condition (iii) in Definition 1.3 is not a characterization, and it merely implies (ii). This can be inferred from work of Khruschev in [14], and this idea is further elaborated on in [24]. Also one might ask what happens if $\alpha \geq 1$ in $(T2)$. This case is less interesting: Volberg's theorem in [29] implies that $\mathcal{P}^2(\mu)$ is then either a close cousin of the Hardy space H^2 (this happens when $\int_{\mathbb{T}} \log w\, dm > -\infty$) or it is not a space of analytic functions at all (if $\int_{\mathbb{T}} \log w, dm = -\infty$). So either way there is no interest in studying this case from the point of view of spaces of analytic functions. See also the introductory section to [23] for a more detailed account.

1.3. **Invariant subspaces generated by singular inner functions.** Having established fairly sharp conditions for irreducibility of our spaces, a way opens to an operator and function theoretic study of this class of spaces. Motivated by certain applications which will soon be detailed, in the first part of the article we study the structure of M_z-invariant subspaces of $\mathcal{P}^2(\mu)$ generated by functions in H^∞, the algebra of bounded analytic functions in $\mathbb{D}$. This question readily reduces to the study of invariant subspaces generated by *singular inner functions*

$$S_\nu(z) = \exp\Big(- \int_{\mathbb{T}} \frac{x+z}{x-z} d\nu(x) \Big), \quad z \in \mathbb{D}, \tag{5}$$

where ν is a finite positive singular Borel measure on $\mathbb{T}$. For $h \in H^\infty$, we will denote by $[h]$ the smallest M_z-invariant subspace containing h, the closure of the linear manifold $\{hp\}_{p\in P}$. It is well-known that any singular inner function generates a non-trivial invariant subspace in the classical Hardy space H^2 of square-summable Taylor series, and it is almost as well-known that in order for S_ν to generate a non-trivial invariant subspace in the standard weighted Bergman spaces (which are $\mathcal{P}^2(\mu)$-spaces of the kind (1) themselves, with $G(t) = t^n$ for some $n > -1$, and $w \equiv 0$) we must have $\nu(A) > 0$ for some Beurling-Carleson set A (see [16], [17], [26]).

Our first main result characterizes the cyclic singular inner functions in the considered class of $\mathcal{P}^2(\mu)$-spaces. By cyclicity we mean that $[S_\nu] = \mathcal{P}^2(\mu)$. It is not hard to see that the minimal considered rate of decay (ExpDec) of the part of μ living on $\mathbb{D}$ makes every non-vanishing bounded function be cyclic in the case that $w = 0$. Thus only properties of w can stop S_ν from being cyclic.

Theorem A. *Let $\mathcal{P}^2(\mu)$ be an irreducible space, in the sense of Definition* 1.3*, defined by a measure μ of the form* (1)*. Then the following two statements are equivalent.*

(i) The singular inner function S_ν is cyclic in $\mathcal{P}^2(\mu)$.

(ii) The measure ν assigns no mass to the core of the weight w:

$$\nu\big(\mathit{core}(w)\big) = 0.$$

Note that $\text{core}(w)$ is open, and hence Borel-measurable, so $\nu\big(\text{core}(w)\big)$ makes perfect sense.

Example 1.4. Let δ_a be a point mass at $a \in \mathbb{T}$, and

$$w(x) = \exp\left(- \frac{1}{|x-1|} \right), \quad x \in \mathbb{T}. \tag{6}$$

Then it is easy to check that

$$\operatorname{core}(w) = \mathbb{T} \setminus \{1\}.$$

Consequently, the singular inner function

$$S_{\delta_a}(z) = \exp\left(-\frac{a+z}{a-z} \right), \quad z \in \mathbb{D}$$

is cyclic, in the considered class of $\mathcal{P}^2(\mu)$ constructed from w appearing in (6), if and only if $a = 1$.

Having settled the cyclicity question, we turn our attention to the invariant subspace $[S_\nu]$ generated by a singular inner function constructed from a singular measure ν which places all its mass on the core: $\nu(\mathbb{T}) = \nu\big(\operatorname{core}(w)\big)$, or in other words when $\operatorname{core}(w)$ is a carrier for ν. A problem which arises in the theory of normalized Cauchy integrals and de Branges-Rovnyak spaces $\mathcal{H}(b)$ (to be discussed below) is to determine which functions are contained in the intersection $H^2 \cap [S_\nu]$, or sometimes in $\mathcal{N}^+ \cap [S_\nu]$, where $\mathcal{N}^+$ is the *Smirnov class* of the disk $\mathbb{D}$ (see [11] for precise definitions):

$$\mathcal{N}^+ = \{u/v : u, v \in H^\infty, \ v \text{ outer}\}$$

In this context, we have the following result.

Theorem B. *Let S_ν be a singular inner function, with defining singular measure ν which satisfies*

$$\nu(\mathbb{T}) = \nu\big(\mathit{core}(w)\big).$$

In an irreducible $\mathcal{P}^2(\mu)$-space defined by a measure μ of the form (1)*, the invariant subspace $[S_\nu]$ satisfies*

$$[S_\nu] \cap \mathcal{N}^+ \subset S_\nu \mathcal{N}^+.$$

In other words, if $f \in \mathcal{N}^+$ can be approximated by polynomial multiples of S_ν in the norm of $\mathcal{P}^2(\mu)$, and ν places all of its mass on $\operatorname{core}(w)$, then S_ν appears in the inner-outer factorization of f. Under the additional assumption that w is bounded, a simple argument will show that in fact $[S_\nu] \cap H^2 = S_\nu H^2$. In [20] and [22], the feature of S_ν appearing in Theorem B is called its *permanence property*. It is obvious that a singular inner function satisfying the permanence property cannot be cyclic.

For the considered class of spaces, Theorem A and Theorem B completely determine the structure of M_z-invariant subspaces generated by bounded analytic functions. Indeed, it follows that if $h = BS_\nu U \in H^\infty$ is the inner-outer factorization of h into a Blaschke product B, singular inner function S_ν and outer function U, then

$$[h] = [S_{\nu_w}],$$

where ν_w is the restriction of the singular measure ν to the set $\operatorname{core}(w)$.

1.4. **Functions of rapid spectral decay and uncertainty relations.** The above results have various applications to the theory of Cauchy integrals, de Branges-Rovnyak spaces, and the following class of functions. This study takes up the second part of the article.

Definition 1.5. (Functions of rapid spectral decay) Let $f(z) = \sum_{n\geq 0} f_n z^n$ be an analytic function in $\mathbb{D}$. If the Taylor coefficients $\{f_n\}_{n\geq 0}$ decay so fast that for some $c > 0$ we have

$$\text{(RSD)} \qquad \sup_{n\geq 0} |f_n| \exp\big(c\sqrt{n}\big) < \infty,$$

then we say that f is a function of *rapid spectral decay.*

Trivial examples of functions f satisfying (RSD) are the analytic polynomials, and functions which extend analytically to a larger disk $r\mathbb{D} = \{z \in \mathbb{C} : |z| < r\}$, $r > 1$. In those cases, the limit in (RSD) is zero even when the term $\exp\big(-c\sqrt{n}\big)$ in (RSD) is replaced by $\exp\big(-cn^\alpha\big)$ for $\alpha \leq 1$. If $\alpha = 1$, then the condition is of course equivalent to f having an analytic extension to some disk around the origin of radius larger than 1.

Let us assume that ν is a positive finite Borel measure for which the Cauchy integral

$$\mathcal{C}_\nu(z) := \int_{\mathbb{T}} \frac{1}{1-\overline{x}z} d\nu(x), \quad z \in \mathbb{D} \tag{7}$$

is a function satisfying (RSD). Can we say something about the nature of the measure ν? As is well known, the Cauchy integral $\mathcal{C}_\nu$ has a representation of the form

$$\mathcal{C}_\nu(z) = \sum_{n\geq 0}^{\infty} \nu_n z^n, \quad z \in \mathbb{D}$$

where $\{\nu_n\}_{n\geq 0}$ is the sequence of Fourier coefficients of ν indexed by non-negative numbers. The rest of the coefficients are annihilated under $\mathcal{C}$, and the condition (RSD) gives us no information about ν_n for $n < 0$. However, plenty of results in harmonic analysis show us how smallness of the spectrum of a function translates into lower bounds on the size, in some sense, of the function itself. The following uncertainty-type statement can be deduced from our results.

Theorem C. *Let ν be a positive finite Borel measure on $\mathbb{T}$, and assume that the Cauchy integral $\mathcal{C}_\nu$, given by* (7)*, satisfies* (RSD)*. Then the measure ν is absolutely continuous with respect to the Lebesgue measure dm:*

$$d\nu = g \cdot dm, \quad g \in \mathcal{L}^1(\mathbb{T}),$$

and there exists an open set U which differs from

$$\{x \in \mathbb{T} : |g(x)| > 0\} \tag{8}$$

only by a set of m-measure zero, with the property that to each $x \in U$ there corresponds an interval I contained in U for which we have

$$\int_I \log |g(x)| \, dm(x) > -\infty.$$

The function $\log |g|$ is, in general, not integrable on the entire open set U appearing in Theorem C.

In a way, Theorem C is similar to the classical theorem of brothers Riesz on structure of measures ν on $\mathbb{T}$ with vanishing positive Fourier coefficients. In our setting, the vanishing of the coefficients is replaced by a weaker condition of their rapid decay forced by the condition (RSD). It should be noted that if we were to replace in (RSD) the term $\exp\left(c\sqrt{n}\right)$ by $\exp\left(cn^\alpha\right)$ for any $\alpha < 1/2$, and thus consider the weaker unilateral spectral decay condition

$$\sup_{n \geq 0} |\nu_n| \exp\left(cn^\alpha\right) < \infty,$$

then a structural result for ν as in Theorem C does not hold: $d\nu = g \cdot dm$ will still be absolutely continuous, but examples show that g can be chosen so that the set in (8) is closed and contains no interval. This follows from a related work of Khrushchev in [14]. There should be room for a slight improvement of the result (see the discussion in Section 1.6.4 below). We ought to mention also that Volberg in [29] found spectral decay conditions making the set in (8) fill up the whole circle $\mathbb{T}$. We will return to both these works below.

1.5. **Condition** (RSD) **in $\mathcal{H}(b)$-spaces.** In most classical Hilbert spaces of analytic functions in the disk, the family of functions which extend analytically to a larger disk form a dense subset of the space. Notable examples include Hilbert spaces of normalized Cauchy integrals. These are the *model spaces*

$$\mathcal{K}_\theta = H^2 \ominus \theta H^2,$$

where θ is an inner function, and the broader class of de Branges-Rovnyak spaces $\mathcal{H}(b)$, where the symbol b is any analytic self-map of the unit disk. There are several ways to define the space $\mathcal{H}(b)$, the easiest perhaps being by stating that it is the Hilbert space of analytic functions on $\mathbb{D}$ with a reproducing kernel of the form

$$k_b(\lambda, z) = \frac{1 - \overline{b(\lambda)}b(z)}{1 - \overline{\lambda}z}, \quad \lambda, z \in \mathbb{D},$$

or alternatively by realizing it as the space of *normalized* Cauchy integrals of functions $\tilde{f} \in \mathcal{L}^2(\nu_b)$, given in the special case $b(0) = 0$ by the formula

$$f(z) = \big(1 - b(z)\big) \int_{\mathbb{T}} \frac{\tilde{f}(x)}{1 - \overline{x}z} d\nu_b(x), \quad z \in \mathbb{D}. \tag{9}$$

Here ν_b is the *Aleksandrov-Clark measure* of b, these two objects being related by the formula

$$\operatorname{Re}\left(\frac{1 + b(z)}{1 - b(z)}\right) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|x - z|^2} d\nu_b(z), \quad z \in \mathbb{D}. \tag{10}$$

The *normalization* refers to multiplication of the Cauchy integral in (9) by the factor $1 - b(z)$, which ensures that the product lands in H^2, as opposed to only being a member of H^p for $p < 1$ (which holds for any Cauchy integral). It is well-known that model spaces $K_\theta = \mathcal{H}(\theta)$ correspond to the purely singular measures ν_θ in (10), and vice versa. In fact, every positive finite Borel measure ν on $\mathbb{T}$ corresponds to a function $b = b_\nu$ through the formula (10). See [5] for more details.

If θ is a singular inner function, then $\mathcal{K}_\theta$ will contain no functions which extend analytically across $\mathbb{T}$. Moreover, it is a consequence of deep results on cyclicity of singular inner functions of Beurling from [3], and also of more recent results of El-Fallah, Kellay and Seip in [8], that in fact if θ is singular inner, then for *any* non-zero function $f(z) = \sum_{n \geq 0} f_n z^n \in \mathcal{K}_\theta$ and for any $c > 0$ it holds that

$$\sup_{n \to \infty} |f_n| \exp\big(cn^{1/2}\big) = +\infty.$$

The breakpoint $\alpha = 1/2$ is sharp: if ν assigns mass to any singleton, then for any $\alpha < 1/2$ and $c > 0$ the space $\mathcal{K}_{S_\nu}$ will contain a non-zero function f satisfying

$$\sup_{n \geq 0} |f_n| \exp\big(cn^{\alpha}\big) < \infty.$$

These facts are not as deep as the two results cited above which imply it. Nevertheless, since they are highly relevant in our context, we will give elementary proofs in Section 6.1. We mention also that a characterization of density in $\mathcal{K}_\theta$ of functions in $\mathcal{A}^\infty$, the algebra of functions analytic in $\mathbb{D}$ with all derivatives extending continuously to $\overline{\mathbb{D}}$, has been established [21].

The situation is more interesting, and much more difficult to handle, in the general class of $\mathcal{H}(b)$-spaces. As in model spaces, it is easy to see that, in general, functions with analytic continuation to a disk larger than $\mathbb{D}$ are not dense in $\mathcal{H}(b)$. It was proved long ago by Sarason that the set $\mathcal{P}$ of analytic polynomials is contained and dense in $\mathcal{H}(b)$ if and only if b is a non-extreme point of the unit ball of H^∞, a condition characterized by

$$\int_{\mathbb{T}} \log(1 - |b|^2)\, dm > -\infty. \tag{11}$$

In terms of the core sets, this result can be stated as follows, and a proof can be found in [27].

Theorem (Sarason). *Let $b : \mathbb{D} \to \mathbb{D}$ be an analytic function, and set*

$$\Delta_b(x) := \sqrt{1 - |b(x)|^2}, \quad x \in \mathbb{T}. \tag{12}$$

The following three statements are equivalent.

(i) The analytic polynomials are dense in $\mathcal{H}(b)$.
(ii) The function b is a non-extreme point of the unit ball of H^∞.
(iii) We have the set equality core$(\Delta_b) = \mathbb{T}$.

Since these conditions are very restrictive, it is tempting to make an effort to capture a larger class of symbols b for which $\mathcal{H}(b)$ contains a dense subset of functions in some nice regularity class which is strictly larger than $\mathcal{P}$. The article [22] connects the approximation problem in $\mathcal{H}(b)$ with the structure of M_z-invariant subspaces of $\mathcal{P}^2(\mu)$, and [20] refines the method to prove the density of $\mathcal{A}^\infty \cap \mathcal{H}(b)$ for a large class of symbols b. By consideration of examples, this class of symbols is close to optimal, but a definitive characterization hinges on the solution of another old conjecture of Kriete and MacCluer from their article [18] on $\mathcal{P}^2(\mu)$-spaces and measures μ of the form (1) where $G(1-|z|)$ is slowly (sub-exponentially) decaying near the boundary. The method from [22] is very general and applies to a wide range of approximation problems in $\mathcal{H}(b)$. In particular, it applies to approximations by functions in the class (RSD). Since our structural results in Theorem A and Theorem B are definitive, we can prove also a definitive result on existence and density of functions $f \in \mathcal{H}(b)$ which satisfy (RSD). In fact, we will prove a much stronger (and optimal) result.

In order to state our result, we will need to quantify the spectral decay of a function f by a condition of the type (RSD) but with $\exp\left(c\sqrt{n}\right)$ replaced by faster increasing sequences. To this end, we define below in Definition 5.1 the *admissible sequences* $M = \{M_n\}_{n\geq 0}$. These sequences are logarithmically convex (at least eventually, for large n) and are decreasing to zero at least as fast as $\exp\big(-c\sqrt{n}\big)$, but satisfy a condition of the form

$$\sum_{n\geq 0} \frac{\log 1/M_n}{1+n^2} < \infty$$

which prohibits, for instance, their exponentially fast decay.

Example 1.6. The sequence defined by

$$M_n = \exp\Bigg(-c\frac{n}{(\log(n)+1)^p}\Bigg), \quad n \geq 1 \tag{13}$$

is admissible for every $p > 1$ and $c > 0$, but it is not admissible for $p = 1$ and any $c > 0$.

We can now state our existence result on functions with rapid spectral decay in $\mathcal{H}(b)$. The quantity Δ_b, appearing in (*iii*) below, was defined in (12).

Theorem D. *Let $b : \mathbb{D} \to \mathbb{D}$ be an analytic function. The following three statements are equivalent.*

(i) The space $\mathcal{H}(b)$ contains a non-zero function f which satisfies (RSD).

(ii) For any admissible sequence $\{M_n\}_{n\geq 0}$, the space $\mathcal{H}(b)$ contains a non-zero function $f(z) = \sum_{n\geq 0} f_n z^n$ which satisfies

$$\sup_{n\geq 0} \frac{|f_n|}{M_n} < \infty \tag{14}$$

(iii) The function b vanishes at some point $\lambda \in \mathbb{D}$, or there exists an arc $I \subset \mathbb{T}$ of positive length for which

$$\int_I \log \Delta_b \, dm > -\infty.$$

In fact, to reach Theorem D we only really need the characterization of irreducibility of $\mathcal{P}^2(\mu)$ from [23] and ideas from [22]. The next theorem pertaining to density, however, requires the full strength of the invariant subspace results developed in the first part of this article.

Theorem E. *Let $b = BS_\nu U$ be the inner-outer factorization of b, and set*

$$\Delta_b(x) = \sqrt{1-|b(x)|^2}, \quad x \in \mathbb{T}.$$

The following three statements are equivalent.

(i) For some $c > 0$, the set of functions f in $\mathcal{H}(b)$ which satisfy

$$\sup_{n\geq 0} |f_n| \exp\Big(c\sqrt{n}\Big) < \infty \tag{15}$$

is dense in $\mathcal{H}(b)$.

(ii) For any admissible sequence $\{M_n\}_{n\geq 0}$, the set of functions f in $\mathcal{H}(b)$ which satisfy

$$\sup_{n\geq 0} \frac{|f_n|}{M_n} < \infty \tag{16}$$

is dense in $\mathcal{H}(b)$.

(iii) The set core(Δ_b) is a carrier for Δ_b and for the singular measure ν.

Example 1.7. For instance, by applying our theorem to the admissible sequence (13) for any $p > 1$, we get that the density in $\mathcal{H}(b)$ of functions $f(z) = \sum_{n\geq 0} f_n z^n$ satisfying

$$\lim_{n\geq 0} |f_n| \exp(cn^\alpha) = 0$$

simultaneously for any $c > 0$ and $\alpha \in (0, 1)$, is equivalent to condition (iii) in Theorem E. Roughly speaking, functions satisfying such decay condition just barely fail to have an analytic continuation to a disk larger than $\mathbb{D}$.

Example 1.8. Generalizing the setting of Example 1.4, we may replace a point by a general closed subset E of $\mathbb{T}$, and define the outer function $b_0 : \mathbb{D} \to \mathbb{D}$ by specifying its modulus $|b_0(x)|$, $x \in \mathbb{T}$, to satisfy (for m-almost every $x \in \mathbb{T}$) the equation

$$\sqrt{1 - |b_0(x)|^2} = \Delta_{b_0}(x) := \frac{1}{2} \exp\left(- \frac{1}{\operatorname{dist}(x, E)} \right), \quad x \in \mathbb{T},$$

where $\operatorname{dist}(x, E)$ is the Euclidean distance from the point x to the closed set E. We can easily check that

$$\operatorname{core}(\Delta_{b_0}) = \mathbb{T} \setminus E.$$

If B is a Blaschke product and S_ν is a singuler inner function, then functions of rapid spectral decay will be dense in the space $\mathcal{H}(b)$, with $b := BS_\nu b_0$, if and only if $\nu(E) = 0$.

Proof of Theorem E given here is very similar to proofs in [20] and [22], but in the present work we obtain some new information on which functions in $\mathcal{H}(b)$ fail to be approximable by classes appearing in Theorem E. These results are presented in Section 7. For instance, a consequence of Proposition 7.3 below is the following: if b_0 is the outer function constructed according to Example 1.8, B is a Blaschke product, S_ν is a singular inner function, and $\nu = \nu_{\mathbb{T}\setminus E} + \nu_E$ is the decomposition of the singular measure ν into pieces living on $\mathbb{T}\setminus E$ and E respectively, then the orthogonal complement of the norm-closure in $\mathcal{H}(b)$ of functions satisfying condition (RSD) is precisely $\widetilde{b} \cdot K_{S_{\nu_E}}$, where $\widetilde{b} = BS_{\nu_{\mathbb{T}\setminus E}} b_0$.

We mentioned earlier that our result is optimal. This is morally true, in the following sense. Assume that $M = \{M_n\}_{n\geq 0}$ is a logarithmically convex sequence which is not admissible according to Definition 5.1, because we have

$$\sum_{n\geq 0} \frac{\log M_n}{1+n^2} = -\infty. \tag{17}$$

For instance, M could be defined by (13) for $p = 1$. If Volberg or Kriete and MacCluer were interested in approximations in $\mathcal{H}(b)$-spaces, they would have proved the following theorem by a use of their techniques in [18] and [29].

Theorem (Volberg, Kriete-MacCluer). *Let $M = \{M_n\}_{n\geq 0}$ be a logarithmically convex sequence satisfying the property* (17)*. The following two statements are equivalent.*

(i) The space $\mathcal{H}(b)$ contains a non-zero function f which satisfies

$$\sup_{n\geq 0} \frac{|f_n|}{M_n} < \infty.$$

(ii) The function b vanishes at some point $\lambda \in \mathbb{D}$, or b is non-extreme.

We do not prove the above theorem in the present article. Its proof is completely analogous to the proof of Theorem D below. The difference consists merely of a use of theorems and observations of Volberg and Kriete-MacCluer from the above mentioned papers, instead of main theorem of [23] as we do here in the proof of Theorem D .

It follows that the investigation of existence and approximability properties in $\mathcal{H}(b)$ of functions with spectral decay satisfying at least (RSD) is essentially completed in Theorem E.

1.6. **Additional comments, questions and conjectures.**

1.6.1. *Work of McCarthy and Davis.* The class of functions satisfying (RSD) has already appeared in the theory of de Branges-Rovnyak spaces. In [7], McCarthy and Davis showed that a function h satisfies (RSD) if and only if the multiplication operator M_h acts boundedly on $\mathcal{H}(b)$ for *all* non-extreme symbols b. In particular, this means that every space $\mathcal{H}(b)$ defined by a non-extreme symbol b contains all functions satisfying (RSD). Our Theorem D then establishes a converse statement: a characterization of b for which $\mathcal{H}(b)$ contains no non-zero such functions.

1.6.2. *Relation to Khrushchev's results.* Khrushchev in [14] studied a problem similar to one appearing in Theorem C. If 1_E is the characteristic function of a set E contained in $\mathbb{T}$, and there exists a function g living only on E such that $\mathcal{C}_g = \mathcal{C}_{g1_E}$ in (7) has some smoothness properties (weaker than our (RSD) condition) then what can be said about E? Khrushchev used the phrase *removal of singularities of Cauchy integrals* in the context of his study of nowhere dense sets E in $\mathbb{T}$ which admit such a multiplier g making $g1_E$ have a smooth Cauchy integral. Thus *"removing"* the singularities of the irregular set E. His solution is given in terms of Beurling-Carleson sets. The weighted version of the problem replaces 1_E by a general weight w.

In turn, Theorem E can be seen as a solution to the problem of removal of singularities of normalized Cauchy integrals in context of the class (RSD), where the possible existence of a singular part of the measure ν forces the normalization. Given a positive finite Borel measure ν on $\mathbb{T}$, we may ask if the space $L^2(\nu)$ contains a dense subset of functions $\tilde{f}$ for which the normalized Cauchy integral in (9) (with ν_b replaced by ν) satisfies (RSD). The condition, in terms of the associated function $b = b_\nu$ given by (10), is explained by (*iii*) of Theorem E. In this context, it would be of interest to characterize intrinsically the measures ν which correspond to b satisfying condition (*iii*) of Theorem E.

Question 1. Let $b : \mathbb{D} \to \mathbb{D}$ be an analytic functions which satisfies the condition (*iii*) in Theorem E. Can we describe the structure of the corresponding Aleksandrov-Clark measure ν_b of b appearing in formula (10) ?

1.6.3. *Logarithmic convexity of admissible sequences.* In spite of some efforts, the author has not been able to remove the assumption of logarithmic convexity in Definition 5.1. Surely the most interesting admissible sequences, such as (13), do satisfy such a conditon, but ideally one would like to remove this assumption. Logarithmic convexity of $\{M_n\}_{n\geq 0}$ plays its part in the proof of Proposition 5.9. In relation to that, we would like to answer the following question.

Question 2. If $c(x)$, $x > 0$, is an increasing, positive and continuous function which satisfies

$$\int_1^\infty \frac{c(x)}{x^2}\, dx < \infty, \tag{18}$$

then under what additional conditons on c may we replace $c(x)$ in (18) by its least concave minorant?

Any interesting condition on c which guarantees the above integrability property of its concave minorant will lead to an improved version of our theorems.

1.6.4. *Non-integrability of* $\log G$ *as a sharp condition.* Consider the condition

$$\int_0^d \log(1/G(x))\, dx < \infty \tag{LogInt}$$

for some $d > 0$. The condition (ExpDec) implies that our considered functions $G(x)$ will always fail to satisfy (LogInt). In fact, it is (at least in the mind of the author) reasonable to conjecture that several of the results of this article should have sharp improvements in which the requirement for G to satisfy (ExpDec) is replaced by the requirement for G *not* to satisfy (LogInt). This condition is, in turn, equivalent to the statement that

$$\sum_{n\geq 0} \frac{\big(\log 1/M_n(G)\big)^2}{1+n^2} = \infty,$$

where $\{M_n(G)\}$ is defined in (39) and is the sequence of moments of the function G. This equivalence can be deduced using techniques appearing in Section 5 below. The above condition appears in [8] as a necessary and sufficient condition for all singular inner functions to be cyclic in a space $\mathcal{P}^2(\mu_{\mathbb{D}})$ with $d\mu(z) = G(1-|z|)dA(z)$, and so $w \equiv 0$ in contrast to the situation dealt with in the present article.

For instance, a sharp version of the irreducibility of $\mathcal{P}^2(\mu)$ with μ of the form (1) would follow if we could prove the following statement.

Conjecture 1. Assume that G fails to satisfy (LogInt) and $w \in \mathcal{L}^1(\mathbb{T})$ is a non-negative weight on $\mathbb{T}$. If $\mathcal{P}^2(\mu)$ of the form (1) is a space of analytic functions on $\mathbb{D}$, then the set $\operatorname{core}(w)$ of a weight $w \in \mathcal{L}^1(\mathbb{T})$ is a carrier for w.

Given this result, one could attempt to combine our techniques appearing in Section 4 and those of El-Fallah, Kellay and Seip from [8] to prove the following strong version of both their result and our Theorem A.

Conjecture 2. In the setting of Conjecture 1, a singular inner function S_ν is cyclic in the space of analytic functions $\mathcal{P}^2(\mu)$ if and only if $\nu\big(\operatorname{core}(w)\big) = 0$.

In relation to Theorem C, we expect the following improvement.

Conjecture 3. The conclusion of Theorem C can be reached if

$$\mathcal{C}_\nu(z) = \sum_{n\geq 0} \nu_n z^n$$

merely satisfies

$$\sup_{n\geq 0} \frac{|\nu_n|}{M_n} < \infty$$

for some (say, logarithmically convex) sequence $\{M_n\}_{n\geq 0}$ satisfying

$$\sum_{n\geq 0} \frac{\big(\log 1/M_n\big)^2}{1+n^2} = \infty.$$

One can show, by considerations of examples, that all of the above conjectures imply sharp results.

1.7. **Outline of the rest of the article.** Section 2 deals with construction of special domains which look like wizard hats and which support very large positive harmonic functions. We prove Theorem B and Theorem A in Sections 3 and 4, respectively. Proof of Theorem B relies heavily on results of Section 2. The second part of the article starts in Section 5, and this section deals with some preparatory estimates on moments sequences which are needed later. We include a detailed account, but most of the results presented in the section are not new, instead being scattered around the existing literature in different versions. Theorem D and Theorem E are proved in the last two Sections 6 and 7. The techniques used in these sections come from [22], but we refine some of the methods and prove auxilliary results of hopefully independent interest. Finally, in Section 8, we prove Theorem C. Section 6 also contains the corresponding results in model spaces which were mentioned above.

1.8. **Some notation.** For a measure μ on $\overline{\mathbb{D}}$ we will denote by $\mu_{\mathbb{D}}$ and $\mu_{\mathbb{T}}$ its restriction to $\mathbb{D}$ and $\mathbb{T} = \partial\mathbb{D}$, respectively. In some contexts we will also use the same notations $\mu_{\mathbb{T}}$ and $\mu_{\mathbb{D}}$ to emphasize that the considered measure lives only $\mathbb{T}$ or $\mathbb{D}$. The area measure dA will always be normalized by the condition $A(\mathbb{D}) = 1$, and a similar convention will be used also for the arc-length measure on the circle: $m(\mathbb{T}) = 1$. We let $\log^+(x) = \max\big(1, \log x\big)$.

The symbol $\|\cdot\|_\mu$ always denotes the usual $\mathcal{L}^2(\mu)$-norm corresponding to the finite positive Borel measure μ. For a set $E \subset \mathbb{T}$, we sometimes use the shorter notation $\mathcal{L}^2(E)$ to denote the space of functions on $\mathbb{T}$ which vanish outside of E and are square-integrable with respect to the Lebesgue measure m. The notation $\big\langle \cdot, \cdot \big\rangle$ denotes different kinds of duality pairings between spaces. By $\big\langle \cdot, \cdot \big\rangle_{\mathcal{L}^2}$ we will denote the standard inner product in $L^2(\mathbb{T})$.

The operator $P_+ : \mathcal{L}^2(\mathbb{T}) \to H^2$ is the orthogonal projection onto the Hardy space H^2. For a bounded analytic function h, the notation $T_{\overline{h}} : H^2 \to H^2$ stands for the co-analytic Toeplitz operator with symbol h, given by formula $T_{\overline{h}} f = P_+ \overline{h} f$.

2. Wizard hats and their harmonic measures

The proof of Theorem B relies on a technique of restriction of a convergent sequence of analytic functions to a certain subdomain of $\mathbb{D}$. It is easier to construct the corresponding domain in the setting of a half-plane, and later use a conformal mapping argument. We will work in the upper half-plane $\mathbb{H}$. There, our domain looks like a wizard's hat (see Figure 1).

Harmonic measures will play an important role in our discussion, so we start by recalling some basic related notions, and set some further notations. Let Ω be a bounded Jordan domain in the complex plane $\mathbb{C}$. The domains Ω which will appear in our context have a boundary $\partial\Omega$ which consists of a finite union of smooth curves. Let $\omega(z, E, \Omega)$ denote the harmonic measure of a segment $E \subset \partial\Omega$ based at the point $z \in \Omega$. Then

$$z \mapsto \omega(z, E, \Omega), \quad z \in \Omega$$

is a positive harmonic function in Ω which extends continuously to the boundary $\partial\Omega$ except at the endpoints of E and possibly finitely many points near which $\partial\Omega$ is not smooth. It attains the boundary value 1 at the relative interior of E, and boundary value 0 on $\partial\Omega \setminus E$, except possibly at the endpoints of the segment E. Let $\mathcal{B}(\partial\Omega)$ denote the Borel σ-algebra on $\partial\Omega$. For each fixed $z_0 \in \Omega$, the mapping

$$A \mapsto \omega(z_0, A, \Omega), \quad A \in \mathcal{B}(\partial\Omega)$$

defines a positive Borel probability measure on $\partial\Omega$. The reader can consult the excellent books by Garnett and Marshall [12] and by Ransford [25] for more background and other basic facts about harmonic measure which are used in this section.

Let

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$$

denote the upper half-plane of $\mathbb{C}$. The main efforts of this section will go into estimation of the harmonic measure on a *wizard hat domain* W. The domain will constructed from an interval $I \subset \mathbb{R}$ and a *profile function* $p(x)$, $x \geq 0$, which by our definition is an increasing, positive and continuous function, smooth (say, continuously differentiable) for $x > 0$, and which satisfies $p(0) = 0$. Given a profile function p and an interval $I = (a, b)$, we define the wizard hat W to be the simply connected Jordan domain

$$W = W(p, I) := \Big\{z = x + iy \in \mathbb{H} : x \in I, y < \min\big[p(x-a), p(b-x)\big]\Big\}. \tag{19}$$

The boundary ∂W is a piecewise smooth curve, with three smooth parts divided by three cusps. An example of a domain W, constructed from a profile function of the type $p(x) = x^q$ for some $q > 1$, is marked by the shaded area in Figure 1.

Our goal is to prove a result regarding existence of harmonic functions which grow rapidly along $\partial W \cap \mathbb{H}$. This growth will be defined in terms of a majorant function F.

Definition 2.1. (Majorants) Let $d > 0$ be some positive number. A positive function $F : (0, d) \to (0, \infty)$ will be called a *majorant* if it satisfies the following two properties:

(i) $F(t)$ is a decreasing function of $t > 0$, and $\lim_{t\to 0^+} F(t) = +\infty$,
(ii) $\int_0^d \log F(t)\, dt < \infty$.

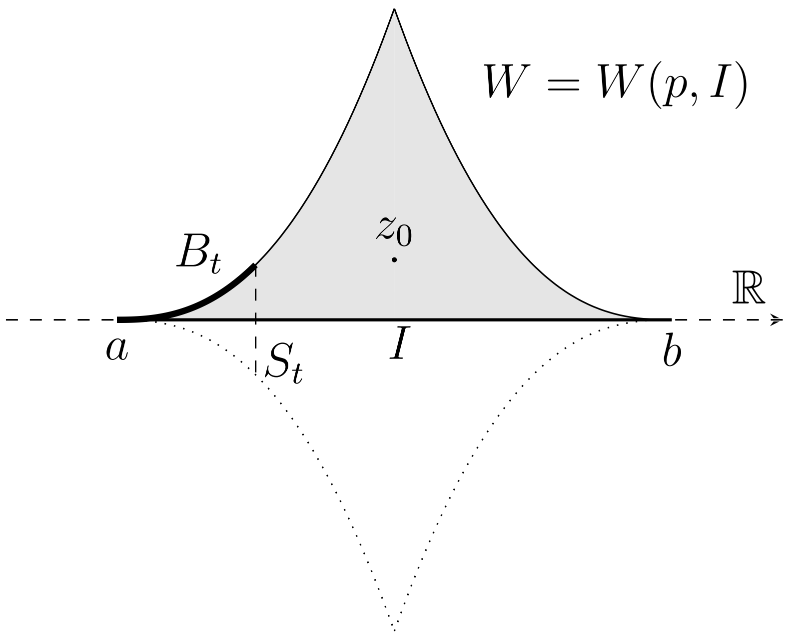

FIGURE 1. The wizard hat W and a piece B_t of its boundary.

The properties of F appearing in Definition 2.1 are related to growth of functions in the investigated class of $\mathcal{P}^2(\mu)$-spaces. See Lemma 3.5 below.

Proposition 2.2. *Let $I \subset \mathbb{R}$ be any finite interval and F be a majorant in the sense of Definition 2.1. Then there exists a profile function p, a wizard hat $W = W(p, I)$, and a positive harmonic function u on W extending continuously to the boundary ∂W except at the three cusps, which satisfies $u(x) = 0$ for $x \in I$, and*

$$u(z) = F(\operatorname{Im} z), \quad z \in (\partial W \cap \mathbb{H}) \setminus c_0,$$

where c_0 is the cusp of ∂W located in $\mathbb{H}$.

In order to prove Proposition 2.2, we will need to estimate the harmonic measure $\omega(z_0, B_t, W)$ of the following piece B_t of the boundary of W:

$$B_t = \Big\{ z = x + iy \in \partial W : 0 < y,\, a < x < t \Big\}. \tag{20}$$

See Figure 1, where B_t is marked. A result of Beurling and Ahlfors (see [12, Theorem 6.1 of Chapter IV]) can be applied to the union of W, I and the reflected domain $\overline{W} = \{\overline{z} : z \in W\}$ to obtain a good estimate for the harmonic measure of B_t.

Proposition 2.3. ***(Beurling-Ahlfors estimate)*** *Let θ be a positive function defined on an interval $(a,b) \subset \mathbb{R}$, and let Ω be the domain*

$$\Omega = \{z = x + iy : |y| < \theta(x),\ a < x < b\Big\}.$$

If $z_0 \in \Omega$ and $S_a = \{z \in \partial\Omega : \operatorname{Re} z = a\}$ is the left vertical part of the boundary of Ω, then

$$\omega(z_0, S_a, \Omega) \leq \frac{8}{\pi} \exp\left(-2\pi \int_a^{\operatorname{Re} z_0} \frac{dx}{\theta(x)} \right).$$

In Figure 1, the symmetrized domain $\widetilde{W} := W \cup I \cup \overline{W}$ is bounded by the top part of the boundary of W and the dotted reflection below the line $\mathbb{R}$. Let $\widetilde{W}_t$ be the domain obtained by cutting $\widetilde{W}$ along the cross-section $S_t = \{z \in \widetilde{W} : \operatorname{Re} z = t\}$ and keeping the right part of the two resulting pieces. Define W_t similarly (so that W_t is the intersection of $\widetilde{W}_t$ with $\mathbb{H}$). Then the Beurling-Ahlfors estimate immediately implies that

$$\omega(z_0, S_t, \widetilde{W}_t) \leq \frac{8}{\pi} \exp\left(-2\pi \int_t^{\operatorname{Re} z_0} \frac{1}{p(x-a)}\, dx \right).$$

By a comparison of the values on ∂W_t of the two harmonic functions $\omega(z, B_t, W)$ and $\omega(z, S_t, \widetilde{W}_t)$, and the maximum principle for harmonic functions, we get the inequality

$$\omega(z, B_t, W) \leq \omega(z, S_t, \widetilde{W}_t), \quad z \in W_t. \tag{21}$$

In particular, this holds at $z_0 \in W$. Hence we have obtained the following harmonic measure estimation.

Proposition 2.4. *Let $W = W(p, I)$ be the wizard hat given by* (19)*, B_t the piece of its boundary given by* (20) *and $z_0 \in W$ be a point on the vertical symmetry line of W. Then*

$$\omega(z_0, B_t, W) \leq \frac{8}{\pi} \exp\left(-2\pi \int_t^{\operatorname{Re} z_0} \frac{1}{p(x-a)}\, dx \right)$$

whenever $a < t < \operatorname{Re} z_0$.

Given a majorant F as in Definition 2.1, we will now show how to construct a profile function p and harmonic function u which satisfy the properties stated in Proposition 2.2. We will assume for simplicity that

$$I = (0, 2)$$

so that $a = 0$ and $b = 2$, and the vertical symmetry line of W is $1 + i\mathbb{R} = \{z = x + iy \in \mathbb{C} : x = 1\}$. It will be easy to see that the proof for the general case is the same as the one given below, and our assumption on I merely simplifies the appearing formulas.

For some large integer $n_0 > 0$, we let

$$\alpha_n := 2^{-n-n_0}, \quad n \geq 1. \tag{22}$$

We define also the sequence

$$\gamma_n := \alpha_n \log F(\alpha_n), \quad n \geq 1 \tag{23}$$

This sequence is positive if the integer n_0 in (22) is chosen large enough. We we will assume this to be satisfied. Next, we make the following simple observation regarding the sequence $\{\gamma_n\}_{n\geq 1}$.

Lemma 2.5. *For any $\epsilon > 0$, there exists an integer $n_0 > 0$ such that, with $\{\alpha_n\}_{n\geq 1}$ defined by (22) and $\{\gamma_n\}_{n\geq 1}$ defined by (23), we have*

$$\sum_{n=1}^{\infty} \gamma_n < \epsilon.$$

Proof. Since $F(t)$ is a majorant, by part (i) of Definition 2.1 we have

$$\begin{aligned} \int_0^{\alpha_1} \log F(t)\, dt &= \sum_{n=1}^{\infty} \int_{\alpha_{n+1}}^{\alpha_n} \log F(t)\, dt \\ &\geq \sum_{n=1}^{\infty} \log F(\alpha_n)(\alpha_n - \alpha_{n+1}) \\ &= \sum_{n=1}^{\infty} \log F(\alpha_n)\alpha_{n+1} \\ &= \frac{1}{2}\sum_{n=1}^{\infty} \gamma_n, \end{aligned}$$

where we used that $\alpha_n - \alpha_{n+1} = \alpha_{n+1} = \alpha_n/2$. Now, by property (ii) in Definition 2.1 we have

$$\lim_{\alpha_1 \to 0^+} \int_0^{\alpha_1} \log F(t)\, dt = 0$$

and so our claim follows. □

We may at this point set n_0 to some value which ensures that

$$\sum_{n=1}^{\infty} \gamma_n < 1/2, \tag{24}$$

or in other words, the sum $\sum_{n=1}^{\infty} \gamma_n$ is less than one quarter of the length of the interval $I = (0, 2)$. Further, we let $\{t_n\}_{n\geq 1}$ be a sequence of positive numbers with

$$t_1 = 1,$$

which tends monotonically to zero and which we shall soon define explicitly in terms of F. In fact, we shall define $\{t_n\}_{n\geq 1}$ by specifying the sequence of differences $\{\Delta t_n\}_{n\geq 1}$, where

$$\Delta t_n := t_n - t_{n+1}, \quad n \geq 1.$$

The differences Δt_n are positive numbers, and $t_2, t_3, \ldots$ will be recursively defined in terms of those differences by the relations

$$t_2 = t_1 - \Delta t_1,\ t_3 = t_2 - \Delta t_2,$$

and so on. In order for so defined sequence $\{t_n\}_{n\geq 1}$ to converge to zero it is necessary and sufficient that

$$\sum_{n=1}^{\infty} \Delta t_n = 1, \tag{25}$$

a requirement which we will later ensure. The profile function p will be chosen so that

$$p(t_n) = \alpha_n, \quad n \geq 1. \tag{26}$$

Given any choice of $\{t_n\}_{n\geq 1}$ satisfying the above named properties, one may readily construct a profile function p which interpolates the data (26). Indeed, since the sequence $\{t_n\}_{n\geq 1}$ is assumed to be monotonically decreasing to zero, the function p can be chosen to be smooth, increasing and positive for $t > 0$, and satisfy $p(0) = 0$.

Thus to each sequence $\{t_n\}_{n\geq 1}$ we may associate a profile function p satisfying (26). A proper choice of this sequence will produce a wizard hat with our desired properties. We let $W = W(p, I)$ be the corresponding wizard hat, and $\omega(\cdot) = \omega(z_0, \cdot, \partial W)$ be the harmonic measure at some $z_0 = 1 + y_0 i \in W$ which lies on the symmetry line of W. Then ω is a Borel probability measure on ∂W. Let $\tilde{u}(z)$ be defined on ∂W by

$$\tilde{u}(z) = \begin{cases} F(\operatorname{Im} z), & z \in \partial W \cap \mathbb{H}, \\ 0, & z \in \partial W \cap \mathbb{R}. \end{cases} \tag{27}$$

We shall show that an appropriate choice of $\{t_n\}_{n\geq 1}$ will ensure that $\tilde{u} \in \mathcal{L}^1(\omega)$. Since F is decreasing, the definition of W ensures that for any $n \geq 1$ the values of the function $\tilde{u}$ on the arc $B_{t_n} \setminus B_{t_{n+1}}$ are dominated by its value at the point $z \in B_{t_n} \setminus B_{t_{n+1}}$ which lies closest to the real line $\mathbb{R}$, i.e., at the point $z = t_{n+1} + ip(t_{n+1})$. In other words, we have

$$\sup_{z \in B_{t_n} \setminus B_{t_{n+1}}} \tilde{u}(z) = F\big(p(t_{n+1})\big). \tag{28}$$

Moreover, from positivity and monotonicity of p, and from Proposition 2.4, we deduce the estimate

$$\omega\big(B_{t_n} \setminus B_{t_{n+1}}\big) \leq \omega(B_{t_n}) \leq \frac{8}{\pi} \exp\left(- 2\pi \int_{t_n}^{t_{n-1}} \frac{1}{p(x)} dx \right)$$

$$\leq \frac{8}{\pi} \exp\left(- 2\pi \frac{\Delta t_{n-1}}{p(t_{n-1})} \right) \tag{29}$$

which holds for $n \geq 2$.

By symmetry of $\tilde{u}$ with respect to the vertical symmetry line of W and the fact that $\tilde{u}$ is unbounded only near the two cusps of ∂W on $\mathbb{R}$, in order to show that $\tilde{u} \in \mathcal{L}^1(\omega)$ it will be sufficient to show that $\int_{B_{t_2}} \tilde{u}(z) d\omega(z) < \infty$. To this end, we use (28) and (29) to estimate

$$\begin{aligned}
\int_{B_{t_2}} \tilde{u}(z) d\omega(z) &= \sum_{n=2}^{\infty} \int_{B_{t_n} \setminus B_{t_{n+1}}} \tilde{u}(z) d\omega(z) \\
&\leq \frac{8}{\pi} \sum_{n=2}^{\infty} F\big(p(t_{n+1})\big) \exp\left(- 2\pi \frac{\Delta t_{n-1}}{p(t_{n-1})} \right) \\
&= \frac{8}{\pi} \sum_{n=2}^{\infty} \exp\left(\log F\big(p(t_{n+1})\big) - 2\pi \frac{\Delta t_{n-1}}{p(t_{n-1})} \right) \\
&= \frac{8}{\pi} \sum_{n=2}^{\infty} \exp\left(\frac{1}{p(t_{n+1})} \Big(\log F\big(p(t_{n+1})\big) p(t_{n+1}) - 2\pi \Delta t_{n-1} \frac{p(t_{n+1})}{p(t_{n-1})} \Big) \right) \\
&= \frac{8}{\pi} \sum_{n=2}^{\infty} \exp\left(2^{n+1+n_0} \Big(\gamma_{n+1} - \frac{\pi \Delta t_{n-1}}{2} \Big) \right).
\end{aligned} \tag{30}$$

In the last step we used (22), (23) and (26). Note specifically that the sequence $\{\gamma_n\}_{n \geq 1}$ in (23) is defined in terms of F and $\{\alpha_n\}_{n \geq 1}$ only, and in particular independently of $\{t_n\}_{n \geq 1}$. We may now specify the values of $\{t_n\}_{n \geq 1}$ by setting the values of the differences:

$$\Delta t_{n-1} = \frac{A}{n^2} + \frac{2}{\pi} \gamma_{n+1}, \quad n \geq 2. \tag{31}$$

for an appropriate constant $A > 0$ which ensures the necessary summation condition (25). This can be done, since

$$\sum_{n=2} \frac{2}{\pi} \gamma_{n+1} < 1/\pi < 1$$

by (24). With these choices we obtain from (30) that

$$\int_{B_{t_2}} \tilde{u}(z)d\omega(z) \leq \frac{8}{\pi}\sum_{n=2}^{\infty} \exp\Big(- \frac{A\pi}{2}\frac{2^{n+1+n_0}}{n^2}\Big) < \infty.$$

Consequently, with this definition of $\{t_n\}_{n\geq 1}$ and a corresponding profile function p, we have that $\tilde{u} \in \mathcal{L}^1(\omega)$.

We may now complete the proof of Proposition 2.2. Let W and p be chosen as above. Let $\{u_n\}_{n\geq 0}$ be a sequence of non-negative continuous functions on ∂W which increase monotonically to the function $\tilde{u}$ given by formula (27). We have above verified that $\tilde{u} \in \mathcal{L}^1(\omega)$. Solve the Dirichlet problem in W with boundary value u_n, and thus extend u_n to W. The maximum principle for harmonic functions then ensures that the sequence $\{u_n(z)\}_{n\geq 0}$ is increasing in n for each $z \in W$. Set

$$u(z) := \lim_{n\to\infty} u_n(z), \quad z \in W.$$

The series converges to a finite value at $z_0 \in W$, since $u_n(z)$ is bounded above by the finite $\mathcal{L}^1(\omega)$-norm of $\tilde{u}$. By the classical Harnack's inequality the limit above exists for each $z \in W$, and u is a positive harmonic function. An elementary argument will show that the function u extends continuously to ∂W except at the three cusps, and it satisfies $u(z) = \tilde{u}(z)$ whenever $z \in \partial W$ is not one of those cusps (for instance, one may adapt parts of the proof of [12, Theorem 1.1 of Chapter II] to see this). Thus from (27) we see that u is the harmonic function sought in Proposition 2.2, and so our proof of that proposition is complete.

3. Proper invariant subspaces generated by singular inner functions

The goal of this section is to prove Theorem B.

3.1. **Technical lemmas.**

Similarly to Section 2, we prove the next lemma in the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. This is done, again, only for convenience. An elementary conformal mapping argument carries the result over to the intended domain $\mathbb{D}$.

In this section, the Lebesgue measure (length measure) on $\mathbb{R}$ will be denoted by dt, and the dt-measure of a set A will be denoted by $|A|$, similar to lengths of sets on the circle $\mathbb{T}$ (this should not cause confusion). The algebra of bounded analytic functions in $\mathbb{H}$ will be denoted by $H^\infty(\mathbb{H})$. In the proofs below we shall also use some basic facts regarding Hardy spaces on $\mathbb{H}$, and in particular some factorization results. An exposition of the relevant background can be found in [15, Chapter VI].

Definition 3.1. (Uniform absolute continuity) If $\{f_n\,dt\}_{n\geq 1}$ is a sequence of non-negative absolutely continuous Borel measures on $\mathbb{R}$ and $I \subset \mathbb{R}$ is an interval, then we will say that the sequence $\{f_n\,dt\}_{n\geq 1}$ is *uniformly absolutely continuous on* I if to each $\epsilon > 0$ there corresponds a $\delta > 0$, independent of n, such that

$$A \subset I,\ |A| < \delta \implies \int_A f_n\,dt < \epsilon.$$

We adopt a similar definition of uniform absolute continuity for families of measures defined on the circle $\mathbb{T}$. Note that in some textbooks, the concept of uniform absolute continuity instead goes under the name *uniform integrability*.

Recall that the class of majorants, appearing in the lemma below, has been introduced in Definition 2.1.

Lemma 3.2. *Let I be a finite interval of the real line $\mathbb{R}$, $\theta = S_\nu$ a singular inner function in $\mathbb{H}$ defined by a singular measure ν supported in the interior of I, and $\{h_n\}_{n\geq 1}$ a sequence of functions in $H^\infty(\mathbb{H})$ such that*

$$\lim_{n\to\infty} \theta(z)h_n(z) = h(z), \quad z \in \mathbb{H},$$

where $h \in H^\infty(\mathbb{H})$ is a non-zero function. Assume that

(i) there exists a majorant F for which we have

$$\sup_{z=x+iy\in R} |\theta(z)h_n(z)| \exp\big(-F(y)\big) < C$$

for some constant $C > 0$ independent of n, and where R is some rectangle in $\mathbb{H}$ with base I:

$$R = R(I,d) := \{z = x+iy \in \mathbb{H} : x \in I, y < d\},$$

(ii) the sequence of positive Borel measures $\{\log^+ |h_n| dt\}_{n\geq 1}$ is uniformly absolutely continuous on I.

Then $h/\theta \in H^\infty(\mathbb{H})$.

Proof. The assumption (ii) implies that

$$\sup_n \int_I \log^+ |h_n|\,dt < \infty.$$

So, denoting by 1_I the characteristic function of the interval I and by passing to a subsequence, we can assume that the measures $1_I \log^+ |h_n| dt$ converge weak-star to a non-negative measure ν supported on I. The measure ν must be absolutely continuous with respect to dt: any set $N \subset I$ of dt-measure zero can be covered by an open set U of total length arbitrarily small, and then we can use (ii) to conclude that

$$\nu(N) \leq \nu(U) = \liminf_{n\to\infty} \int_U \log^+ |h_n| dt < \epsilon$$

for any $\epsilon > 0$. Consequently $\nu(N) = 0$ whenever N is a dt-null set. So, by the classical Radon-Nikodym theorem, we have that $d\nu = w\, dt$ for some non-negative $w \in \mathcal{L}^1(I)$. We denote by u_I the harmonic function in $\mathbb{H}$ which is the Poisson extension of the measure $d\nu = w\, dt$ to $\mathbb{H}$:

$$u_I(z) = \frac{1}{\pi} \int_I \frac{y}{(x-t)^2 + y^2} w(t) dt, \quad z = x + iy \in \mathbb{H}.$$

Let also u_n denote the Poisson extension of the measure $1_I \log^+ |h_n| dt$:

$$u_n(z) = \frac{1}{\pi} \int_I \frac{y}{(x-t)^2 + y^2} \log^+ |h_n(t)|\, dt, \quad z = x + iy \in \mathbb{H}.$$

The assumption (i) implies that

$$\log |\theta(z)| + \log |h_n(z)| \leq A + F(y), \quad z = x + iy \in R$$

for some positive constant $A > 0$. Consequently, by Proposition 2.2, there exists a wizard hat domain $W = W(p, I)$ and a positive harmonic function u defined on W which extends continuously to the boundary ∂W except at three cusps, $u(x) = 0$ for x in the interior of I, and which satisfies

$$\log |\theta(z)| + \log |h_n(z)| \leq u(z), \quad z \in \partial W \cap R.$$

But by the assumption that the singular measure defining θ is supported in the interior of I, it follows that θ is analytic and non-zero in a neighbourhood of $\partial W \cap R$, and so the quantity $\log |\theta(z)|$ is bounded from above and from below on $\partial W \cap R$. Therefore, by possibly replacing u by a positive scalar multiple of itself, in fact we have that

$$\log |h_n(z)| \leq u(z), \quad z \in \partial W \cap R.$$

For the bottom side I of the wizard hat, we instead have that for dt-almost every z in I, non-tangential limit of the subharmonic function $\log |h_n(z)|$ is dominated by the non-tangential limit of $u_n(z)$. This follows immediately from classical boundary behaviour properties of Poisson integrals. We would like to conclude that these two inequalities imply

$$\log |h_n(z)| \leq u(z) + u_n(z), \quad z \in W. \tag{32}$$

Indeed such a generalization of the maximum principle for subharmonic functions holds, and we will carefully verify this claim in Lemma 3.3 below. Assuming the claim, we recall that

$$h_n(z) \to h(z)/\theta(z), \quad n \to +\infty$$

in all of $\mathbb{H}$, and so by letting $n \to +\infty$ we obtain, from (32) and the earlier mentioned weak-star convergence of measures (which guarantees that $u_n(z) \to u(z)$ for $z \in \mathbb{H}$), that

$$\log|h(z)| - \log|\theta(z)| \leq u(z) + u_I(z), \quad z \in W. \tag{33}$$

Since $h \in H^\infty(\mathbb{H})$, we have a representation

$$\begin{aligned} \log|h(z)| = & \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2}\log|h(t)|dt \\ & - \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2}d\nu_h(t) \\ & + \sum_{h(\alpha)=0} \log\Big(\frac{|z-\alpha|}{|z-\overline{\alpha}|}\Big), \quad z = x+iy \in \mathbb{H} \end{aligned} \tag{34}$$

where ν_h is a positive singular Borel measure. We also have a similar representation for $\log|\theta|$:

$$\log|\theta(z)| = -\frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2}d\nu(t). \tag{35}$$

Let J be some interval containing the support of ν, and which is strictly contained in I. Well-known properties of Poisson integrals, the fact that the zero set of h must satisfy a Blaschke condition, and formula (34), imply together that as the positive number y tends to zero, the restrictions to J of the real-valued measures $\log|h(t+iy)|dt$ converge weak-star to the restriction to J of the measure $\log|h(t)|dt - d\nu_h(t)$. Similar claim holds for $\log|\theta|$ and its defining measure ν, and for the harmonic function u_I and $w\,dt$.

For sufficiently small $y > 0$ we have that $J + iy := \{x + iy : x \in J\} \subset W$. Thus from the weak-star convergence of measures discussed above, the inequality (33), and the fact that $u \equiv 0$ on J, we obtain the real-valued measure inequality

$$\log|h(t)|dt - d\nu_h(t) + d\nu(t) \leq w(t)dt \quad \text{ on } J.$$

This measure inequality is to interpreted in the following way:

$$w(t)dt - \log|h(t)|dt + d\nu_h(t) - d\nu(t)$$

is a non-negative measure on J. The dt-singular part of this measure is $d\nu_h - d\nu$, which is thus non-negative on J. Since $d\nu$ is supported inside J, in fact $d\nu_h - d\nu$ is non-negative in all of $\mathbb{R}$. Now subtracting (35) from (34) and using the non-negativity of $d\nu_h - d\nu$, we get

$$\log|h(z)| - \log|\theta(z)| \leq \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2}\log|h(t)|dt$$
$$+ \sum_{h(\alpha)=0} \log\Big(\frac{|z-\alpha|}{|z-\overline{\alpha}|}\Big), \quad z = x+iy \in \mathbb{H}$$

and finally by exponentiating, we obtain

$$|h(z)/\theta(z)| \leq \|h\|_{H^\infty(\mathbb{H})}, \quad z \in \mathbb{H}.$$

□

We need to verify the claim made in the proof of Lemma 3.2 which lead to the fundamental inequality (32).

Lemma 3.3. *Let the wizard hat $W = W(p, I)$ be as in the proof of Lemma 3.2, f be a bounded analytic function in W and u be a positive harmonic function in W. Assume that both f and u extend continuously to $\partial W \cap \mathbb{H}$ and also that both have non-tangential limits almost everywhere on I. If we have that $\log|f(z)| \leq u(z)$ for $z \in \partial W \cap \mathbb{H}$, and moreover that the non-tangential limits of $\log|f|$ on I are dt-almost everywhere dominated by the non-tangential limits of u on I, then $\log|f(z)| \leq u(z)$ for all $z \in W$.*

Proof. Let $\phi : \mathbb{D} \to W$ be a conformal mapping. The local smoothness of the boundary of W and basic conformal mapping theory ensure that ϕ is conformal at almost every point of $\mathbb{T}$ (see [12, Chapter V.5]). This implies that the functions $\log|f \circ \phi|$ and $u \circ \phi$, which are defined in $\mathbb{D}$, have non-tangential limits almost everywhere on $\mathbb{T}$. Let $\widetilde{u \circ \phi}$ be a harmonic conjugate of $u \circ \phi$ in $\mathbb{D}$ and consider the function $H(z) = \exp\Big(- u \circ \phi(z) - i\widetilde{u \circ \phi}(z)\Big)$, $z \in \mathbb{D}$. By positivity of u, the function H is bounded in $\mathbb{D}$, and our assumptions leads to the conclusion that the non-tangential boundary values on $\mathbb{T}$ of the bounded function $(f \circ \phi)(z)H(z) \in H^\infty(\mathbb{D})$ are not larger than 1 in modulus. Thus by basic function theory in $\mathbb{D}$, we obtain the inequality $|(f \circ \phi)(z)H(z)| \leq 1$ for all $z \in \mathbb{D}$. This easily translates into $\log|f(z)| \leq u(z)$ for $z \in W$. □

We will need Lemma 3.2 in the disk $\mathbb{D}$. Here is the precise statement which we will use.

Corollary 3.4. *Let I be an arc the circle $\mathbb{T}$, $\theta = S_\nu$ a singular inner function in $\mathbb{D}$ defined by a singular measure ν supported in the interior of I, and $\{h_n\}_{n\geq 1}$ a sequence of functions in H^∞ such that*

$$\lim_{n\to+\infty} \theta(z)h_n(z) = h(z), \quad z \in \mathbb{D}$$

where $h \in H^\infty$. Assume that

(i) there exists a majorant F for which we have

$$\sup_{z \in \mathbb{D}} |\theta(z) h_n(z)| \exp\Big(- F\big(1 - |z|\big)\Big) < C$$

for some positive constant $C > 0$ independent of n,

(ii) the sequence of positive Borel measures $\{\log^+ |h_n| dm\}_{n \geq 1}$ is uniformly absolutely continuous on I.

Then $h/\theta \in H^\infty$.

It is easy to see that Lemma 3.2 implies Corollary 3.4. Indeed, if $\phi : \mathbb{H} \to \mathbb{D}$ is a conformal map for which $\phi^{-1}(I)$ is a finite segment on $\mathbb{R}$, then the distortion of lengths and distances by the map ϕ is bounded above and below near $\phi^{-1}(I)$ and I, since ϕ is a bi-Lipschitz bijection between some open sets containing $\phi^{-1}(I)$ and I. Thus, for instance, the growth condition (i) in Corollary 3.4 for the sequence $\{S_\nu h_n\}_{n \geq 1}$ is easily translated into a corresponding condition (i) in Lemma 3.2 for the sequence $\{S_{\tilde{\nu}} \tilde{h}_n\}_{n \geq 1}$, where $S_{\tilde{\nu}} := S_\nu \circ \phi$ and $\tilde{h}_n := h_n \circ \phi$, by replacing $F(t)$ with a new majorant of the form $\tilde{F}(t) := F(at)$ for some $a > 0$. Moreover, the mapping ϕ will transform a uniformly absolutely continuous family of measures on $\phi^{-1}(I)$ into a uniformly absolutely continuous family on I. Thus Corollary 3.4 can readily be deduced from Lemma 3.2 by a change of variables argument. We shall leave out further details of this straight-forward but somewhat tedious computation.

We note in passing that Corollary 3.4 is in fact a vast generalization of one of the main results of [19] related to permanence properties of singular inner functions under weak convergence in H^p-spaces.

3.2. **Proof of Theorem** B**.** Theorem B follows almost immediately from Corollary 3.4, we just need to verify that a bounded sequence in the corresponding $\mathcal{P}^2(\mu)$-space satisfies properties (i) and (ii) in Corollary 3.4. This is done in the next two lemmas.

Lemma 3.5. *Let*

$$d\mu_{\mathbb{D}}(z) = G(1 - |z|) dA(z),$$

where G satisfies the condition (LogLogInt) *appearing in Section 1. For $z \in \mathbb{D}$, denote by*

$$E_z := \sup_{\substack{f \in \mathcal{P}^2(\mu_{\mathbb{D}}) \\ f \neq 0}} \frac{|f(z)|}{\|f\|_{\mu_{\mathbb{D}}}}$$

the norm of the evaluation functional $z \mapsto f(z)$ on $\mathcal{P}^2(\mu_{\mathbb{D}})$. There exists a majorant F such that

$$E_z \leq \exp\big(F(1 - |z|)\big), \quad z \in \mathbb{D}.$$

Proof. Fix $z \in \mathbb{D}$, $\delta = (1-|z|)/2$ and let $B(z,\delta)$ denote the ball around z of radius δ. By subharmonicity of the function $z \mapsto |f(z)|$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |f(z)| &\leq \frac{1}{\delta^2} \int_{B(z,\delta)} |f(z)| dA(z) \\ &\leq \frac{1}{\delta^2} \|f\|_{\mu_{\mathbb{D}}} \sqrt{\int_{B(z,\delta)} \frac{1}{G(1-|z|)} dA(z)}. \end{aligned}$$

Since G is assumed to be an increasing function, we may estimate the integral inside the square root by

$$\int_{B(z,\delta)} \frac{1}{G(1-|z|)} dA(z) \leq \frac{1}{\delta^2} \frac{1}{G\big((1-|z|)/2\big)}.$$

Putting this into the above estimates, we obtain

$$|f(z)| \leq \frac{1}{\delta^3} \sqrt{\frac{1}{G\big((1-|z|)/2\big)}}.$$

Now set

$$F(t) := \log\left(\frac{8}{t^3}\right) + \frac{1}{2}\log\left(\frac{1}{G(t/2)}\right), \quad t \in (0,1].$$

By the above estimate, the norm E_z of the evaluation functional is bounded by

$$E_z \leq \exp\big(F(1-|z|)\big), \quad z \in \mathbb{D}.$$

Moreover, F is a decreasing function, and by virtue of G satisfying (LogLogInt), F also certainly satisfies

$$\int_0^d \log F(t)\, dt < \infty$$

if $d > 0$ is some small number. Thus F is a majorant in the sense of Definition 2.1. □

Lemma 3.6. *Assume that the weight w satisfies*

$$\int_I \log w\, dm > -\infty$$

for some arc $I \subset \mathbb{T}$. If $\{f_n\}_{n\geq 1}$ are positive functions such that

$$\int_I f_n^p w\, dm < C, \quad n \geq 1$$

for some constant $C > 0$ and some $p > 0$, then the sequence $\{\log^+ f_n\, dt\}_{n\geq 1}$ is uniformly absolutely continuous on I.

Proof. Note that

$$\begin{aligned}\log^+ f_n &\leq \log^+(f_n w^{1/p}) + \log^+(w^{-1/p}) \\ &\leq \frac{1}{p}\log^+(f_n^p w) + \frac{1}{p}\log^+(1/w) \\ &:= g_n + g,\end{aligned}$$

where it follows from the assumption that g_n are positive functions which form a bounded subset of (say) $\mathcal{L}^2(I)$, and g is in $\mathcal{L}^1(I)$. Clearly, if A is a measurable subset of I, then by Cauchy-Schwarz inequality we obtain

$$\int_A g_n \, dm \leq \sqrt{A} \cdot \|g_n\|_{\mathcal{L}^2(I)},$$

so that the family $\{g_n dm\}_{n\geq 1}$ is uniformly absolutely continuous on I. Then the above inequalities imply that $\{\log^+ f_n \, dt\}_{n\geq 1}$ is a uniformly absolutely continuous sequence on I. □

Proof of Theorem B. Let $h \in [S_\nu] \cap \mathcal{N}^+$. Since $h \in [S_\nu]$, there exists a sequence of polynomials $\{p_n\}_{n\geq 1}$ such that $S_\nu \cdot p_n$ converges to h in the norm of $\mathcal{P}^2(\mu)$. Multiplying h by a suitable bounded outer function u we can ensure that hu is bounded, i.e., $hu \in H^\infty$, and that $S_\nu p_n u$ converges to hu. Let $\{K_j\}_j$ be an increasing sequence of compact sets such that $\cup_j K_j = \operatorname{core}(w)$. By Lemma 3.5, Lemma 3.6 and Corollary 3.4, whenever ν_j is the restriction of ν to the compact subset K_j, we have that $hu/S_{\nu_j} \in H^\infty$, with the bound $\|hu/S_{\nu_j}\|_\infty \leq \|hu\|_\infty$ following from elementary Hardy space theory. The assumption that $\nu(\mathbb{T}) = \nu\big(\operatorname{core}(w)\big)$ means that the restrictions $\{\nu_j\}_j$ converge weak-star, as $j \to \infty$, to the measure ν. Thus

$$|h(z)u(z)/S_\nu(z)| = \lim_{j\to\infty} |h(z)u(z)/S_{\nu_j}(z)| \leq \|hu\|_\infty, \quad z \in \mathbb{D}$$

In particular, since u is outer, it follows that S_ν divides the inner factor of h. Thus $h/S_\nu \in \mathcal{N}^+$. □

4. Cyclic singular inner functions

In this section we will study the cyclicity of singular inner functions, and prove Theorem *A*. The method which we will employ is a much simpler variant of the argument appearing in [23] which establishes the equivalence between the different conditions in Definition 1.3. The essence of the proof in [23] is the construction of a certain sequence of real-valued functions with very particular oscillation properties. Our current task will require a similar construction, but without the requirement of oscillation.

4.1. **Weak-star approximation of singular measures, with obstacles.** Let ν be a positive singular Borel measure on $\mathbb{T}$. The cyclicity in $\mathcal{P}^2(\mu)$ of the singular inner function S_ν corresponding to the singular measure ν will follow from the existence of a sequence of non-negative bounded functions $\{f_n\}_{n\geq 1}$ for which the measures $\{f_n \, dm\}_{n\geq 1}$ converge weak-star to ν. Such a sequence of course always exists, and is easy to construct, but in our context the functions f_n will have to satisfy a severe restriction on their size, namely

$$0 \leq f_n(x) \leq \log^+(1/w(x)), \quad x \in \mathbb{T}. \tag{36}$$

In case $w(x) = 0$, the right-hand side is to be interpreted as $+\infty$ (i.e., no size restriction on f_n at the point x).

Essentially, the obstacle (36) prohibits the existence of an approximating sequence $\{f_n dm\}_{n\geq 1}$ if some part of the mass of ν is located in "wrong" places on $\mathbb{T}$ where w is large. However, if ν is carried outside of the core set of w, then such a sequence exists. This is the content of the next lemma.

Lemma 4.1. *Let ν be a positive singular Borel measure which satisfies*

$$\nu\big(core(w)\big) = 0.$$

Then there exists a sequence of bounded functions $\{f_n\}_{n\geq 1}$ satisfying the following three properties:

(i) the non-negative measures $\{f_n \, dm\}_{n\geq 1}$ converge weak-star to ν,
(ii) $\int_{\mathbb{T}} f_n \, dm = \nu(\mathbb{T})$,
(iii) the functions f_n obey the bound (36).

Proof. Let us first suppose that ν assigns no mass to any singletons, so that $\nu(\{x\}) = 0$ whenever $x \in \mathbb{T}$. For any positive integer n, we let D_n be the family of 2^n disjoint open dyadic intervals, each of length $2\pi \cdot 2^{-n}$, such that their union covers the circle $\mathbb{T}$ up to finitely many points, and such that the system $\cup_{n\geq 1} D_n$ possesses the usual dyadic nesting property: each $d \in D_n$ is contained in a unique $d' \in D_{n-1}$. Fixing an integer $n \geq 1$, we will specify how to define f_n on each of the intervals $d_j \in D_n$, $1 \leq j \leq 2^n$, in such a way that the above three properties hold.

If $\nu(d_j) = 0$, then we simply set $f_n \equiv 0$ on d_j. Conversely, if $\nu(d_j) > 0$, then since $\nu\big(\text{core}(w)\big) = 0$, it must be that $\nu(d_j) = \nu(d_j \setminus \text{core}(w)) > 0$. It follows that the set $d_j \setminus \text{core}(w)$ must be non-empty. Pick some point $x \in d_j \setminus \text{core}(w)$. It follows from the definition of core(w) that for any open interval I which contains x in its interior we have $\int_I \log w \, dm = -\infty$ (else x would have been a member of core(w)). Since $w \in \mathcal{L}^1(\mathbb{T})$, it must be so that $\int_I \log^+(1/w) \, dm = +\infty$. Pick such an interval I which is contained within

d_j. If there exists a subset $A \subset I$ satisfying $m(A) = |A| > 0$ on which w is identically zero, then we may set

$$f_n(x) = \nu(d_j)|A|^{-1}1_A(x), \quad x \in d_j$$

where 1_A is the characteristic functions of A. In case such a set does not exist, then $w > 0$ almost everywhere on I, we proceed to define f_n on d_j in a different way. In this case, we must have

$$+\infty = \int_I \log^+(1/w)\, dm = \lim_{c \to 0^+} \int_{I \cap \{w > c\}} \log^+(1/w)\, dm$$

so that

$$\nu(d_j) < \int_{I \cap \{w > c\}} \log^+(1/w)\, dm < +\infty$$

for all sufficiently small $c > 0$. By absolute continuity of the finite measure

$$\log^+(1/w) 1_{I \cap \{w > c\}} dm$$

there must then exist a set $B \subset I \cap \{w > c\}$ for which we have precisely

$$\nu(d_j) = \int_B \log^+(1/w)\, dm$$

We pick such a set B and define

$$f_n(x) = \log^+(1/w(x)) 1_B(x), \quad x \in d_j.$$

We can set f_n to be equal to zero outside of $\cup_{j=1}^{2^n} d_j$. One way or the other, we have defined f_n as a bounded function, and we have

$$\int_{d_j} f_n\, dm = \nu(d_j).$$

By summing this over all the 2^n intervals d_j, we see that property (ii) in the statement of the lemma is satisfied (since ν assigns no mass to the finitely many points outside the union of the open intervals d_j). Property (iii) is satisfied by the construction. Property (i) also holds. Indeed, if g is the characteristic function of one of the dyadic intervals from some stage of our construction, then the nesting property of the dyadic system and the additivity of ν ensure that

$$\lim_{n \to \infty} \int_{\mathbb{T}} g f_n\, dm = \nu(d_j) = \int_{\mathbb{T}} g\, d\nu.$$

The above equalities hold also for functions g which are finite linear combinations of characteristic functions of dyadic intervals. Since such linear combinations can be used to uniformly approximate any continuous function on $\mathbb{T}$, and since we have the uniform variation bound in (ii), we conclude that $f_n\, dm$ converges weak-star to ν. The proof is complete in the case that ν assigns no mass to singletons.

In the contrary case we have that

$$\nu = \sum_j w_j \delta_{x_j}, \quad w_j > 0$$

is a countable linear combination of unit masses δ_{x_j} at the sequence of points $\{x_j\}_j$ in $\mathbb{T}$. Our assumption implies that $x_j \not\in \operatorname{core}(w)$ for all j. Thus each x_j is the midpoint of an interval I which can be chosen to have arbitrarily small length and for which we have $\int_I \log^+(1/w)\, dm = +\infty$. We can then proceed in an analogous way to the above, and produce at each stage n of the construction a disjoint sequence of intervals $\{I_{n,j}\}_j$ each covering a different point x_j to which ν assigns positive mass. We then define a positive function f_n which carries appropriate amount of mass on each of the intervals $I_{n,j}$ and satisfies the other needed properties. We skip laying out the straight-forward details of this adaptation of the previous argument.

The general case follows by decomposing a measure ν into a sum of one measure which is a sum of pure point masses and one measure which assigns no mass to singletons, and then applying the above arguments to these two pieces separately. □

4.2. **Proof of Theorem** A**.** We will need one more elementary lemma. It appears in [8] and many other works.

Lemma 4.2. *Assume that H is a Banach space of analytic functions in $\mathbb{D}$ with the property that for all functions h in H^∞ the operator $M_h f := hf$ is bounded on H. Then the product uv of two cyclic bounded functions $u, v \in H^\infty$ is cyclic.*

By cyclicity of u we mean, of course, that there exists a sequence of analytic polynomials $\{p_n\}_{n\geq 1}$ such that up_n converges to $1 \in H$ in the norm of the space.

Proof. If u and v are two cyclic bounded functions, then for any polynomials p, q we have the inequality

$$\|1 - puv\|_H \leq \|1 - qv\|_H + \|M_v\| \|q - pu\|_H,$$

where $\|M_v\|$ denotes the operator norm of the multiplication operator M_v, and $\|\cdot\|_H$ denotes the norm in H. We use cyclicity of v and choose the polynomial q to make the first term on the right arbitrarily small, and next we use cyclicity of u and choose p to make the second term on the right arbitrarily small. It follows that the product uv of two bounded cyclic functions is a cyclic function. □

Lemma 4.2 applies to any irreducible space $\mathcal{P}^2(\mu)$ of the form considered here, since indeed the multiplication by any bounded analytic function induces a bounded operator on these spaces. We skip the straight-forward proof, which can for instance be based on simple analysis of the dilations $h_r(z) := h(rz)$, $r \in (0,1)$, of the bounded function h. In particular, the proposition implies that $H^\infty \subset \mathcal{P}^2(\mu)$, whenever the latter is irreducible. For future reference, note that as a subspace of $\mathcal{L}^2(\mu)$ (with μ as in (1)), each function $h \in H^\infty$ is defined also on $\mu_{\mathbb{T}} := w\, dm$, the part of μ living on the circle $\mathbb{T}$. It is not hard to see that the values of h with respect to $\mu_{\mathbb{T}}$ coincide with its usual non-tangential boundary values of h on $\mathbb{T}$. If w is bounded, then the same conclusions hold also for any $h \in H^2$.

Proof of Theorem A. Note first that (i) certainly implies (ii) in Theorem A, since the condition $\nu\big(\text{core}(w)\big) > 0$ implies that a factor in S_ν satisfies the permanence property exhibited in Theorem B, and so cannot by cyclic. Thus it suffices for us to show the implication $(ii) \Rightarrow (i)$. The norms induced by measures μ satisfying (ExpDec) are largest if the measure μ has the form in $(T1)$, and $\beta = 1$. If S_ν is cyclic in $\mathcal{P}^2(\mu)$ defined by any measure this form, then it is cyclic in any $\mathcal{P}^2(\mu)$-space considered in this article. Thus it suffices to prove the theorem in the case of μ being of the form $(T1)$ with $\beta = 1$, and any $c > 0$.

Let us then assume that $\nu\big(\text{core}(w)\big) = 0$. The formula (5) shows that

$$S_\nu(z) = \prod_{i=1}^{N} S_{\nu/N}(z), \quad z \in \mathbb{D}$$

for any positive integer $N \geq 1$. Then by replacing ν by ν/N for N sufficiently large, and by Lemma 4.2, we may assume that $\nu(\mathbb{T}) < c/10$. Let $\{f_n\}_{n\geq 1}$ be a sequence of positive bounded functions given by Lemma 4.1 for which the measures $\{f_n\, dm\}_{n\geq 1}$ converge weak-star to 2ν, which satisfy $\int_{\mathbb{T}} f_n\, dm = 2\nu(\mathbb{T})$ and for which the bound (36) holds. Construct the outer functions

$$h_n(z) := \exp\Big(H_{f_n}(z)/2\Big), \quad z \in \mathbb{D},$$

where

$$H_{f_n}(z) := \int_{\mathbb{T}} \frac{x+z}{x-z} f_n(x)\, dm(x), \quad z \in \mathbb{D}$$

is the usual Herglotz integral of f_n. Then, since $|H_{f_n}(z)| \leq \frac{4\nu(\mathbb{T})}{1-|z|}$, we obtain

$$|h_n(z)| \leq \exp\Big(\frac{2\nu(\mathbb{T})}{1-|z|}\Big) \leq \exp\Big(\frac{c}{5(1-|z|)}\Big), \quad z \in \mathbb{D},$$

and from property (*iii*) in Lemma 4.1 and basic properties of Herglotz integrals, we have the non-tangential boundary value estimate

$$|h_n(x)| \leq \exp\Big(f_n(x)/2\Big) \leq \sqrt{\max[1, 1/w(x)]}$$

for almost every $x \in \mathbb{T}$ with respect to m. It follows from these inequalities and the definition of the norm in $\mathcal{P}^2(\mu)$ that the family $\{h_n\}_{n\geq 1} \subset H^\infty$ forms a bounded subset of the Hilbert space $\mathcal{P}^2(\mu)$. Moreover, by the weak-star convergence of $\{f_n\, dm\}_{n\geq 1}$ to 2ν we have that

$$\lim_{n\to\infty} h_n(z) = \frac{1}{S_\nu(z)}, \quad z \in \mathbb{D}.$$

But this means that $1/S_\nu$ is a member of $\mathcal{P}^2(\mu)$, since we can identify it as a weak cluster point of some subsequence of $\{h_n\}_{n\geq 1}$. Thus there must also exist a sequence of polynomials $\{p_n\}_n$ tending to $1/S_\nu$ in the norm of $\mathcal{P}^2(\mu)$. Consequently, since M_{S_ν} is a bounded operator on our space, we have that $S_\nu p_n$ tend to 1 in the norm of $\mathcal{P}^2(\mu)$. That is, S_ν is cyclic. □

5. Moment functions, admissible sequences and spaces of Taylor series

This section, together with the following two, constitutes the second part of the article. In this part, we will apply our previous results in $\mathcal{P}^2(\mu)$-theory to problems in the theory Cauchy integrals, model spaces and the de Branges-Rovnyak spaces $\mathcal{H}(b)$. In order to do so, we will need to analyze the moments of the functions G appearing in (1). This entire section is concerned with this analysis.

5.1. Admissible sequences and their properties.

If G is a function satisfying (ExpDec) and (LogLogInt), then the sequence of moments of G, defined below in (39), will be later shown to satisfy the following basic properties, which we summarize in a definition.

Definition 5.1. (Admissible sequences) A decreasing sequence of positive numbers $\{M_n\}_{n\geq 0}$ with

$$\lim_{n\to\infty} M_n = 0$$

will be called *admissible* if it satisfies the following three conditions:

(i) the sequence $\{\log M_n\}_{n\geq 0}$ is eventually logarithmically convex, in the sense that

$$2\log M_n \leq \log M_{n+1} + \log M_{n-1}$$

for all sufficiently large $n \geq 0$,

(ii) there exists $d > 0$ such that

$$M_n \leq \exp(-d\sqrt{n})$$

for all sufficiently large $n \geq 0$,

(iii) the summability condition

$$\sum_{n\geq 0} \frac{\log(1/M_n)}{1+n^2} < \infty$$

is satisfied.

Example 5.2. A simple way to construct an admissible sequence is to set

$$M_n = \exp\big(-c(n)\big),$$

where c is any positive function of $x > 0$ which has a concave restriction to some interval $x \in (N, \infty)$, grows faster than $d\sqrt{x}$ for some $d > 0$, and which satisfies $\int_1^\infty c(x)x^{-2}\, dx < \infty$. Thus, for instance, the sequence $\{M_n\}_{n\geq 0}$ given by

$$M_n = \exp\left(-\frac{dn}{(\log(n)+1)^p}\right), \quad n \geq 0,$$

which was already mentioned in the introduction, is admissible whenever $p > 1$ and $d > 0$.

With later applications in mind, it will be useful to single out the following simple preservation property of admissible sequences under taking powers.

Proposition 5.3. *If $M = \{M_n\}_{n\geq 0}$ is an admissible sequence, then so is*

$$M^p := \{M_n^p\}_{n\geq 0},$$

for any $p > 0$.

The proposition follows immediately from Definition 5.1

5.2. **Legendre envelopes.** Roughly speaking, admissible sequences $\{M_n\}_{n\geq 0}$ are in a correspondence with moments of functions G satisfying (ExpDec) and (LogLogInt), and we shall now proceed to make this statement more precise. In order to do so, we will need to recall some basic concepts from convex analysis. In particular we will use the notion of Legendre envelopes and their properties. In parts of our exposition we will follow Beurling in [4] and Havin-Jöricke in [13].

Let $m(x)$ be a positive and continuous function defined for $x > 0$, which is decreasing and satisfies

$$\lim_{x\to 0^+} m(x) = +\infty.$$

In our application, m will be of the form $m(x) = \log 1/G(x)$ (for small x). The *lower Legendre envelope* m_* is defined as

$$m_*(x) := \inf_{y>0} m(y) + xy, \quad x > 0. \tag{37}$$

Being an infimum of concave (actually affine) and increasing functions, m_* is itself concave and increasing, and it is easy to see that

$$\lim_{x \to +\infty} m_*(x) = +\infty.$$

Remark 5.4. We single out the following important remark. Assume that we modify the function m above for $x > 1$, so that we end up with a different function $\widetilde{m}$ which satisfies $\widetilde{m}(x) = m(x)$ for $x < 1$, but the values of the two functions might differ for $x \geq 1$. Then it is not hard to see from the definition in (37) that $\widetilde{m}_*(x) = m_*(x)$ for all sufficiently large x. This is true since the infimum at the right-hand side of (37) must then, in both the case of m_* and $\widetilde{m}_*$, be attained at some small $y \in (0,1)$. Indeed, if $y > 1$, then we have by positivity of m that

$$m(y) + xy \geq x \geq m(1/2) + x/2,$$

the second inequality holding if x is sufficiently large. Thus for sufficiently large x, the candidate $y = 1/2$ is always better than any candidate $y > 1$ in the infimum in (37), and our claim follows.

In [4, Lemma 1], Beurling proves the following statement which will be used below.

Proposition 5.5. *Let $m(x)$ be a positive, continuous and decreasing function of $x > 0$ which satisfies $\lim_{x \to 0^+} m(x) = +\infty$. The following two statements are equivalent:*

(i) there exists a $\delta > 0$ such that

$$\int_0^\delta \log m(x)\, dx < \infty,$$

(ii) we have

$$\int_1^\infty \frac{m_*(x)}{1+x^2} dx < \infty.$$

We refer the reader to [4] for a proof of Proposition 5.5.

Let k be function which satisfies the same properties as m_*. Namely, let $k(x)$ be a positive concave function of $x > 0$ which is increasing and satisfies

$$\lim_{x \to +\infty} k(x) = +\infty.$$

We will consider its *upper Legendre envelope* defined as

$$k^*(x) := \sup_{y > 0} k(y) - xy. \tag{38}$$

Then it is easy to see that k^* is a convex and decreasing function, and

$$\lim_{x \to 0^+} k^*(x) = +\infty.$$

Thus k^* satisfies properties similar to m above (in addition, it is convex). We have the following inversion formula, which is well-known.

Lemma 5.6. *Let k be a positive concave function of $x > 0$ which is increasing and satisfies* $\lim_{x\to\infty} k(x) = +\infty$. *Then*

$$(k^*)_*(x) = k(x).$$

Proof. Note first that for any $x > 0$ and $y > 0$, we have from (38) that

$$k^*(x) \geq k(y) - xy,$$

so that

$$k^*(x) + xy \geq k(y).$$

Taking the infimum in $x > 0$ of the right-hande side of the above expression tells us, in accordance with (37), that $(k^*)_*(y) \geq k(y)$ for all $y > 0$. For the reverse inequality, recall that concavity of k means that $k(y)$ is the infimum of the values $ay+b$ over pairs $(a, b) \in \mathbb{R}^2$ such that the line $\{at + b : t > 0\}$ lies above the graph of k. Since k is increasing to $+\infty$, for any such pair we must have $a > 0$, and clearly

$$at + b \geq k(t) \quad \Leftrightarrow \quad b \geq k(t) - at, \quad t > 0,$$

which by taking supremum in t and using definition (38) means that $b \geq k^*(a)$. Given $\epsilon > 0$, let (a, b) be such that $ay + b \leq k(y) + \epsilon$ and the line $\{at + b : t > 0\}$ lies above the graph of k. We have just showed that $b \geq k^*(a)$. Therefore

$$\begin{aligned} k(y) + \epsilon &\geq ay + b \\ &\geq ay + k^*(a) \\ &\geq \inf_{a>0} k^*(a) + ay \\ &= (k^*)_*(y), \end{aligned}$$

and so the proof is completed by letting ϵ tend to zero. □

5.3. **A characterization of admissible sequences.** Let μ be a measure of the form (1). We will be interested in the sequence $\{M_n\}_{n\geq 0}$ of moments of the part of μ living on $\mathbb{D}$:

$$\begin{aligned} M_n = M_n(G) &:= \int_{\mathbb{D}} G(1 - |z|)|z|^{2n} dA(z) \\ &= 2\int_0^1 G(1-r)r^{2n+1}dr. \end{aligned} \tag{39}$$

For this reason, to each increasing function G on $[0,1]$ satisfying $G(0)=0$ we associate its *moment function* defined by

$$P_G(x) := \int_0^1 G(1-r)r^x\,dx, \quad x>0. \tag{40}$$

The next lemma gives an estimate on P_G.

Lemma 5.7. *Let $G(x)$, $x\in[0,1]$, be an increasing continuous function satisfying $G(0)=0$, and further let*

$$m(x) := \begin{cases} \log 1/G(x), & x\in(0,1] \\ \log 1/G(1), & x>1. \end{cases} \tag{41}$$

Then, for sufficiently large $x>0$, we have the estimates

$$\frac{\exp\big(-m_*(2x)\big)}{4x} \le P_G(x) \le \exp\big(-m_*(x)\big),$$

where P_G is the moment function of G defined in (40)*.*

Proof. We shall only prove the easier upper estimate for P_G. The lower estimate is established in the course of the proof of [23, Lemma 4.3]. A very similar estimate is also proved in [13, Page 229].

For the upper estimate, we simply use the inequality $\log r\le r-1$ which holds for all $r>0$, and the definition of m_* in (37), to obtain

$$\begin{aligned} P_G(x) &= \int_0^1 \exp\big(\log G(1-r) + x\log r\big)dr \\ &\le \int_0^1 \exp\big(\log G(1-r) - x(1-r)\big)dr \\ &= \int_0^1 \exp\big(-m(t)-xt\big)dt \\ &\le \int_0^1 \exp\big(-m_*(x)\big)dt \\ &= \exp\big(-m_*(x)\big). \end{aligned}$$

□

By (ExpDec), every considered function G satisfies the inequality

$$G(x)\le \exp\Big(-\frac{c}{x}\Big), \quad x\in(0,1)$$

for some constant $c > 0$. Such an inequality of course puts an upper bound on the rate of decrease of the moments of G. We will need the following simple consequence of Lemma 5.7 in order to verify property (ii) of Definition 5.1. The result will also be useful in coming sections.

Proposition 5.8. *For $c > 0$ and $\beta > 0$, let $\{M_n(\beta, c)\}_{n \geq 0}$ be the sequence of moments given by*

$$\begin{aligned} M_n(\beta, c) &:= \int_{\mathbb{D}} \exp\Big(- c(1-|z|)^{-\beta} \Big) |z|^{2n} \, dA(z) \\ &= 2 \int_0^1 \exp\Big(- c(1-r)^{-\beta}\Big) r^{2n+1} \, dr, \quad n \geq 0. \end{aligned} \tag{42}$$

Let

$$\tilde{\beta} := \frac{\beta}{\beta+1}.$$

For any $\delta > 0$ and sufficiently large positive n, we have the estimates

$$\exp\Big(- (d_0(\beta, c) + \delta) n^{\tilde{\beta}} \Big) \leq M_n(\beta, c) \leq \exp\Big(- d_1(c, \beta) n^{\tilde{\beta}}\Big) \tag{43}$$

for some positive functions d_0 and d_1 which satisfy the asymptotics

$$\lim_{c \to +0} d_0(\beta, c) = \lim_{c \to +0} d_1(\beta, c) = 0$$

and

$$\lim_{c \to +\infty} d_0(\beta, c) = \lim_{c \to +\infty} d_1(\beta, c) = +\infty.$$

Proof. We will use Lemma 5.7 and the bounds derived there. In the notation of that lemma, and with

$$G(x) = \exp\Big(\frac{c}{x^\beta}\Big), \quad x \in (0,1)$$

we have

$$m(x) = \frac{c}{x^\beta}, \quad x \in (0,1)$$

and we need to compute the corresponding Lagrange envelope m_* defined in (37). Having fixed some number $x > 0$, we use elementary calculus to show that

$$\inf_{y > 0} \frac{c}{y^\beta} + xy$$

is attained at the point

$$y_* = \left(\frac{c\beta}{x}\right)^{\frac{1}{\beta+1}}$$

from which it follows that

$$m_*(x) = cy_*^{-\beta} + xy_* = d(\beta, c)x^{\tilde{\beta}}$$

for some quantity $d(\beta, c)$ which is readily computed explicitly and which is seen to satisfy the asymptotics that are postulated for d_0 and d_1 in the statement of the proposition. Since

$$M_n(\beta, c) = 2P_G(2n+1)$$

we obtain from Lemma 5.7 the inequalities

$$2\exp\big(-d(\beta,c)(4n+2)^{\tilde{\beta}} - \log(8n+4)\big) \leq M_n(\beta,c) \leq 2\exp\big(-d(\beta,c)(2n+1)^{\tilde{\beta}}\big) \tag{44}$$

which hold for all sufficiently large n. Our result follows easily from this. □

We can now prove the main result of the section, which connects our considered class of functions G with the admissible sequences appearing in Definition 5.1.

Proposition 5.9. *If G satisfies* (ExpDec) *and* (LogLogInt)*, then $\{M_n\}_{n\geq 0}$ defined by*

$$M_n := \int_0^1 G(1-r)r^{2n+1}\,dr \tag{45}$$

is an admissible sequence.

Conversely, if $\{M_n\}_{n\geq 0}$ is an admissible sequence, then there exists a continuous and increasing function G satisfying (ExpDec)*,* (LogLogInt)*, $G(0) = 0$, and for which the inequality*

$$P_G(2n+1) \leq M_n$$

holds for all sufficiently large $n \geq 0$.

Proof. We start by proving that the sequence in (45) is admissible by verifying the three required properties in Definition 5.1. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} M_n &= \int_0^1 G(1-r)r^{(n-1)+1/2}r^{(n+1)+1/2}\,dr \\ &\leq \sqrt{M_{n-1}}\sqrt{M_{n+1}} \end{aligned}$$

It follows easily from this that $\{\log M_n\}_{n\geq 0}$ is a convex sequence. The inequality $M_n \leq \exp(-c\sqrt{n})$ for some $c > 0$ and all sufficiently large $n \geq 0$ follows readily from (ExpDec) and an application of the upper estimate Proposition 5.8 with $\beta = 1$ (and consequently

$\tilde{\beta} = 1/2)$. Let m be as in Lemma 5.7. By that lemma, we have

$$\sum_{n\geq 0} \frac{\log 1/M_n}{1+n^2} = \sum_{n\geq 0} \frac{-\log P_G(2n+1)}{1+n^2}$$
$$\leq \sum_{n\geq 0} \frac{m_*(4n+2) + \log(8n+4)}{1+n^2}.$$

Now the assumption that G satisfies (LogLogInt) implies that $\int_0^1 \log m(x)\, dx < \infty$, and so from Proposition 5.5 we deduce easily that the last sum above is convergent. Consequently, $\{M_n\}_{n\geq 0}$ is an admissible sequence.

Conversely, assume that $\{M_n\}_{n\geq 0}$ is any admissible sequence. Since the sequence tends to zero, we may without loss of generality assume that $M_0 < 1$. From property (i) in Definition 5.1 we easily deduce the inequality

$$\log 1/M_{n+1} - \log 1/M_n \leq \log 1/M_n - \log 1/M_{n-1}, \quad n \geq 0.$$

This means that the slopes of the line segments between each consecutive pair of points in the sequence

$$(2n+1, \log 1/M_n), \quad n \geq 0 \tag{46}$$

are decreasing, which means that if we define the function $k(x)$, $x > 0$, as the piecewise linear interpolant of the data (46), then k is concave, continuous, positive and increasing, and satisfies

$$k(2n+1) = \log 1/M_n, \quad n \geq 0.$$

It also satisfies $\lim_{x\to\infty} k(x) = +\infty$, and property (iii) in Definition 5.1 easily implies that

$$\int_1^\infty \frac{k(x)}{1+x^2}\, dx < \infty. \tag{47}$$

Let k^* be the upper Legendre envelope of k defined in (38), and set

$$G(x) := \exp\big(-k^*(x)\big), \quad x \in (0,1], \tag{48}$$

and $G(0) = 0$. Then G is a continuous and increasing function. Let m be defined as in the statement of Lemma 5.7. The functions k^* and m agree for $x \in (0,1)$, and consequently by our Remark 5.4, the functions $(k^*)_*$ and m_* coincide for large $x > 0$. By the inversion formula in Lemma 5.6 we have the equality $k = (k^*)_*$, and so the functions k and m_* coincide for large x. Define P_G as in (40). Then by Lemma 5.7 and our derived equality between k and m_* we have the estimate

$$P_G(x) \leq \exp\big(-k(x)\big)$$

for all sufficiently large x. Consequently,

$$P_G(2n+1) \leq M_n$$

if n is large, since k interpolates the data (46). Moreover, the equalities $m(x) = k^*(x)$ for $x \in (0,1)$ and $k(x) = (k^*)_*(x) = m_*(x)$ for large $x > 0$, together with (47) and Proposition 5.5, imply that

$$\int_0^1 \log\log 1/G(x)\,dx = \int_0^1 \log m(x)\,dx < \infty.$$

Thus G defined in (48) satisfies (LogLogInt). It remains to check that G also satisfies (ExpDec). Note that property (ii) in Definition 5.1 of the admissible sequence $\{M_n\}_{n\geq 0}$ implies easily that the interpolant k defined above satisfies a lower bound of the form

$$k(x) \geq d\sqrt{x}, \quad x \geq 0$$

for some constant $d > 0$. But then, by (38), we have

$$\begin{aligned} k^*(x) &= \sup_{y\geq 0} k(y) - xy \\ &\geq \sup_{y>0} d\sqrt{y} - xy \\ &= \frac{d^2}{4x}. \end{aligned}$$

The last equality can be derived by elementary calculus techniques. Consequently

$$\liminf_{x\to 0^+} x \log 1/G(x) \geq \frac{d^2}{4} > 0,$$

and so G satisfies (ExpDec). The proof is complete. □

5.4. **Some auxiliary spaces of Taylor series.** If $f : \mathbb{D} \to \mathbb{C}$ is an analytic function and

$$d\mu_{\mathbb{D}}(z) = G(1-|z|)dA(z) \tag{49}$$

then we have the norm equality

$$\|f\|^2_{\mu_{\mathbb{D}}} = \int_{\mathbb{D}} |f(z)|^2 d\mu_{\mathbb{D}}(z) = \sum_{n\geq 0} |f_n|^2 M_n(G) \tag{50}$$

where $\{f_n\}_{n\geq 0}$ is the sequence of Taylor coefficients of f, and $M = \{M_n(G)\}_{n\geq 0}$ is the sequence of moments corresponding to the measure $\mu_{\mathbb{D}}$ and which was defined in (39). The above equality gives us an isometric isomorphism between $\mathcal{P}^2(\mu_{\mathbb{D}})$ and a space of Taylor series $H_2(M)$ which we will find very convenient.

For a decreasing sequence $M = \{M_n\}_{n\geq 0}$ of positive numbers we define $H_2(M)$ as the Hilbert space of analytic functions in $\mathbb{D}$ consisting of $f(z) = \sum_{n\geq 0} f_n z^n$ which satisfy

$$\|f\|^2_{H_2(M)} := \sum_{n\geq 0} M_n |f_n|^2 < \infty. \tag{51}$$

The formal dual space $H_2^*(M)$ is to consist of power series which instead satisfy

$$\|f\|^2_{H_2^*(M)} := \sum_{n\geq 0} \frac{|f_n|^2}{M_n} < \infty. \tag{52}$$

Since $M = \{M_n\}_{n\geq 0}$ is assumed to be decreasing, the space $H_2^*(M)$ is always contained in the Hardy space H^2. In fact, if M is the moment sequence of any function G appearing in our study, then $H_2^*(M)$ consists of functions with very rapidly decaying Taylor series. In particular, the members $f \in H_2^*(M)$ have a continuous extension from $\mathbb{D}$ to $\overline{\mathbb{D}}$, and the restriction of f to the circle T is a smooth function.

The duality between $H_2(M)$ and $H_2^*(M)$ is realized by the usual Cauchy pairing

$$\langle f, g \rangle := \lim_{r\to 1^-} \sum_{n\geq 0} r^{2n} f_n \overline{g_n} = \int_{\mathbb{T}} f\overline{g}\, dm \tag{53}$$

where the sequential definition above makes sense whenever $f \in H_2(M)$, $g \in H_2^*(M)$, and the integral definition holds in special cases, for instance when $f, g \in H^2$-

In our applications, the sequences M will be the admissible sequences studied in Section 5. Such sequences have the property that

$$\lim_{n\to\infty} M_n^{1/n} = 1,$$

a condition which ensures that the spaces $H_2(M)$, and their duals, are genuine spaces of analytic functions on $\mathbb{D}$.

We will also find it convenient to have introduced the space $H_1^*(M)$ which consists of power series $f(z) = \sum_{n\geq 0} f_n z^n$ satisfying

$$\|f\|_{H_1^*(M)} := \sup_{n\geq 0} \frac{|f_n|}{M_n} < \infty. \tag{54}$$

The notation $H_1^*(M)$ is justified by the fact that this space is the formal dual of a space $H_1(M)$ in which the norm is as in (51) but without the squares. Recall from Proposition 5.3 that the family of admissible sequences introduced in Definition 5.1 is invariant under taking powers. For this reason, the instances of spaces $H_2^*(M)$ and $H_1^*(M)$ which appear in our study are in fact very similar.

Lemma 5.10. *Let $M = \{M_n\}_{n\geq 0}$ be an admissible sequence, and for $p > 0$ consider the sequences*

$$M^p := \{M_n^p\}_{n\geq 0}.$$

For $p > 1/2$, we have the continuous embeddings

$$H_1^*(M^p) \subset H_2^*(M) \subset H_1^*(M^{1/2}).$$

Proof. If $f \in H_2^*(M)$, then for any $n \geq 0$ we have that

$$\frac{|f_n|^2}{M_n} \leq \|f\|^2_{H_2^*(M)},$$

so clearly $f \in H_1^*(M^{1/2})$. Conversely, if $f \in H_1^*(M^p)$ for some $p > 1/2$, then we may use that M satisfies property *(ii)* of admissible sequences in Definition 5.1 to obtain

$$\begin{aligned}\sum_{n\geq 0} \frac{|f_n|^2}{M_n} &\leq \|f\|^2_{H_1^*(M^p)} \sum_{n\geq 0} M_n^{2p-1} \\ &\leq \sum_{n\geq 0} \exp\big(-c(2p-1)\sqrt{n}\big) \\ &< \infty.\end{aligned}$$

Thus $f \in H_2^*(M)$. □

We end the section with a few words on operators acting on the introduced class of spaces. From their definition, and in particular from the assumption on M being decreasing, it is not hard to see that the spaces $H_2(M)$ are invariant under the multiplication operator M_z, and that this operator is a contraction on the space. Then Von Neumann's inequality ([1]) or the Sz.-Nagy Foias H^∞-functional calculus ([28]) shows that in fact every function $h \in H^\infty$ defines a bounded multiplication operator $M_h : H_2(M) \to H_2(M)$. The adjoint operator $M_h^* : H_2^*(M) \to H_2^*(M)$ is easily indentified with the usual Toeplitz $T_{\overline{h}}$ operator with the co-analytic symbol $\overline{h}$, i.e., $T_{\overline{h}} f$ is the orthogonal projection to the Hardy space H^2 of the function $\overline{h} f \in \mathcal{L}^2(\mathbb{T})$.

For later reference, we record these observations in a proposition.

Proposition 5.11. *Let $M = \{M_n\}_{n\geq 0}$ be an admissible sequence.*

(i) The space $H_2(M)$ is invariant for the analytic multiplication operators

$$M_h f(z) = h(z) f(z), \quad f \in H_2(M),$$

with symbols $h \in H^\infty$.

(ii) The space $H_2^*(M)$ is invariant for the co-analytic Toeplitz operators

$$T_{\overline{h}}f(z) = P_+\overline{h}f(z), \quad f \in H_2^*(M)$$

with symbols $h \in H^\infty$.

6. Existence in $\mathcal{H}(b)$ of functions with rapid spectral decay

This section deals with proving Theorem D. In the proof, we will need a corresponding result in the context of model spaces. We establish this result first. Next, we present some background theory of $\mathcal{H}(b)$-spaces which will be needed in the proof of Theorem D, and also in the proof of Theorem E, which will be given in the next section.

6.1. **Corresponding results in model spaces.** The following Proposition 6.1 needed in the proof of Theorem D is essentially known, has been mentioned in the introduction, and follows for instance from the work of Beurling in [3], or from a result of El-Fallah, Kellay and Seip in [8]. The mentioned results are much stronger than Proposition 6.1 below. However, because the result is important for our further purposes, we shall use the moment estimates from Section 5 to give a simple proof of our version of the result.

Proposition 6.1. *If θ is a singular inner function, then the model space $\mathcal{K}_\theta$ contains no non-zero function $f(z) = \sum_{n\geq 0} f_n z^n$ which satisfies*

$$\sup_{n\geq 0} |f_n| \exp\left(cn^{1/2}\right) < \infty \tag{55}$$

for any $c > 0$.

Proof. Assume that θ is singular and $f \in \mathcal{K}_\theta$ satisfies (55) for some constant $c > 0$. We will show that $f(0) = f_0 = 0$. Since $\mathcal{K}_\theta$ is invariant for the backward shift

$$Lf(z) := \frac{f(z) - f(0)}{z}, \quad z \in \mathbb{D},$$

and Lf is easily seen to satisfy (55) also, it follows sequentially that $f_1 = 0$, $f_2 = 0$, and so on, for all $n \geq 0$. Thus $f \equiv 0$, and the claim of the proposition will follow.

Condition (55) and Lemma 5.10 imply that f is a member of the space $H_2^*(M)$ defined in Section 5.4, where $M = \{M_n\}_{n\geq 0}$ and

$$M_n = \exp(-dn^{1/2})$$

for some $d > 0$. Let $\mu_{\mathbb{D}}$ be given by the formula

$$d\mu_{\mathbb{D}}(z) = \exp\Big(-\frac{c'}{(1-|z|)}\Big)dA(z).$$

with $c' > 0$ so small that in Proposition 5.8 we have $d > d_0(1, c') + \delta$, and consequently $M_n \leq M_n(1, c')$, which implies a contractive embedding of $\mathcal{P}^2(\mu_{\mathbb{D}})$ inside $H_2(M)$. The measure $\mu_{\mathbb{D}}$ is of the form $(T1)$ for $\beta = 1$ and $w \equiv 0$. By Theorem A, the singular inner function θ is trivially cyclic in $\mathcal{P}^2(\mu_{\mathbb{D}})$, since $\text{core}(w) = \text{core}(0) = \varnothing$. Thus there exists a sequence of analytic polynomials $\{p_n\}_{n\geq 0}$ such that $\theta p_n \to 1$ in the norm of $\mathcal{P}^2(\mu)$. By the above mentioned embedding, we also have convergence in the norm of $H_2(M)$. Thus, using the duality pairing (53) and the membership of f in $H_2^*(M) \cap K_\theta$, the following computation is justified:

$$\begin{aligned} f(0) &= \langle f, 1 \rangle \\ &= \lim_{n\to\infty} \langle f, \theta p_n \rangle \\ &= \lim_{n\to\infty} \int_{\mathbb{T}} f \overline{\theta p_n} \, dm \\ &= 0 \end{aligned}$$

the last equality holding due to f being a member of $K_\theta = (\theta H^2)^\perp$. Thus $f(0) = 0$, and the proof is complete by the initial remarks. □

Proposition 6.1 is interesting it its own right because it exhibits a gap in the possible rate of decay of Taylor series of functions in model spaces. Indeed, either the inner function θ is singular, in which case Proposition 6.1 applies, or θ vanishes at some point $\lambda \in \mathbb{D}$, in which case *exponential decay*

$$|f_n| \leq \exp(-cn)$$

of Taylor coefficients is satisfied for the kernel function $f(z) = k_\lambda(z) := (1 - \overline{\lambda} z)^{-1} \in \mathcal{K}_\theta$.

In fact the exponent $\alpha = 1/2$ in Proposition 6.1 is sharp, in the sense that for any given $\alpha \in (0, 1/2)$ and $c > 0$ one can construct a singular inner function θ and a space $\mathcal{K}_\theta$ which contains non-zero functions satisfying

$$|f_n| \leq \exp\Big(- cn^\alpha \Big).$$

The result is not necessary to prove either Theorem D or Theorem E, but we will take a small detour to establish it. The technique in the proof is similar to the one appearing in Section 3.

Proposition 6.2. *Let θ be the singular inner function corresponding to a point mass at $1 \in \mathbb{T}$:*

$$\theta(z) := \exp\Big(- a \frac{1+z}{1-z} \Big), \quad a > 0.$$

Then, for any fixed choice of $c > 0$ and $\alpha \in (0, 1/2)$, the space $\mathcal{K}_\theta$ contains a non-zero function $f(z) = \sum_{n\geq 0} f_n z^n$ which satisfies

$$\sup_{n\geq 0} |f_n| \exp(cn^\alpha) < \infty.$$

The strategy of the proof is as follows. Let $\beta \in (0, 1)$. If a singular inner function θ is not cyclic in some space $\mathcal{P}^2(\mu_\mathbb{D})$ with measure $\mu_\mathbb{D}$ given by

$$d\mu_\mathbb{D}(z) = \exp\Big(- \frac{1}{(1 - |z|)^\beta} \Big) dA(z) \tag{56}$$

then there must exist a non-zero function $g(z) = \sum_{n\geq 0} g_n z^n \in \mathcal{P}^2(\mu_\mathbb{D})$ such that

$$\sum_{n\geq 0} M_n(\beta, 1) g_n \overline{h_n} = 0$$

whenever $h(z) = \sum_{n\geq 0} h_n z^n$ is the power series of a function of the type $= \theta p$, p being an analytic polynomial. This says that the non-zero power series

$$\tilde{g}(z) = \sum_{n\geq 0} \tilde{g}_n z^n := \sum_{n\geq 0} M_n(\beta, 1) g_n z^n$$

lies in the model space $\mathcal{K}_\theta$, and moreover we have

$$M_n(\beta, 1)^{-1} |\tilde{g}_n|^2 = M_n(\beta, 1) |g_n|^2 \leq \|g\|^2_{\mu_\mathbb{D}}.$$

By Proposition 5.8 we obtain the estimate

$$|\tilde{g}_n| \leq \|g\|_{\mu_\mathbb{D}} \cdot \exp\Big(- d_1(\beta, 1) n^{\tilde{\beta}} \Big) \tag{57}$$

for all sufficiently large n. Here $\tilde{\beta} \in (0, 1/2)$. If the non-cyclicity of θ in $\mathcal{P}^2(\mu)$ is independent of the parameter $\beta \in (0, 1)$, then given any $\alpha \in (0, 1/2)$ we may choose β so close to 1 that the inequality $\tilde{\beta} > \alpha$ holds, and consequently obtain that $\tilde{g} \in \mathcal{K}_\theta$ satisfies the Taylor series decay in Proposition 6.2.

Thus it is sufficient for us to establish the following simple claim.

Proposition 6.3. *For any $\beta \in (0, 1)$, the singular inner function*

$$\theta = \theta_a := \exp\Big(- a \frac{1 + z}{1 - z} \Big), \quad a > 0.$$

is not cyclic in $\mathcal{P}^2(\mu_\mathbb{D})$ constructed from the measure (56).

Again, this result is known and is essentially a simpler version of a more general and more difficult result contained in [8, Theorem 2]. We proceed to give an elementary proof of our version.

We will need the following elementary lemma. The proof only requires a routine argument involving subharmonicity of $|f|$, and so we skip it (see also the proof of Lemma 3.5 above).

Lemma 6.4. *For a given $c > 0$ and $\beta > 0$, let*

$$d\mu_{\mathbb{D}}(z) = \exp\left(- \frac{c}{(1-|z|)^\beta}\right) dA(z).$$

Then there exists a constant $C(\beta, c) > 0$ such that

$$|f(z)| \le C(\beta, c)\|f\|_{\mu_{\mathbb{D}}} \exp\left(\frac{2c}{(1-|z|)^\beta}\right), \quad z \in \mathbb{D}.$$

Proof of Proposition 6.3. Assume, seeking a contradiction, that there exists a sequence of analytic polynomials $\{p_n\}_{n\ge 0}$ such that $\theta p_n \to 1$ in the norm of $\mathcal{P}^2(\mu_{\mathbb{D}})$, with $\mu_{\mathbb{D}}$ of the form (56) and $\beta \in (0,1)$. Then Lemma 6.4 implies that the estimate

$$\begin{aligned} \log|\theta(z)| + \log|p_n(z)| &= -a\frac{1-|z|^2}{|1-z|^2} + \log|p_n(z)| \\ &\le A + \frac{2c}{(1-|z|)^\beta} \end{aligned}$$

holds for all $z \in \mathbb{D}$, where $A > 0$ is some constant. We re-write this as

$$\log|p_n(z)| \le A + \frac{2c}{(1-|z|)^\beta} + 2a\frac{1-|z|}{|1-z|^2}. \tag{58}$$

For $p > 1$, let $\gamma = \gamma(p)$ be the curve

$$\gamma = \{z \in \mathbb{D} : |1-z|^p = (1-|z|)\}.$$

This curve encloses the segment $(0,1) \subset \mathbb{D}$. A quick computation shows that the estimate (58) on the curve γ becomes

$$\log|p_n(z)| \le A + \frac{2c}{|1-z|^{p\beta}} + \frac{2a}{|1-z|^{2-p}}. \tag{59}$$

Since $\beta \in (0,1)$, surely we can choose $p > 1$ close enough to 1 so that $p\beta < 2-p := s < 1$. For such a choice of s we can use (59) to easily obtain

$$\log|p_n(z)| \le \frac{C_0}{|1-z|^s}, \quad z \in \gamma \tag{60}$$

where $C_0 > 0$ is some constant. Using appropriate branch of the holomorphic square root, the function

$$z \mapsto \frac{1}{(1-z)^s}, \quad \operatorname{Re} z > 0,$$

takes its values in an infinite cone centered at the origin, with the positive part of the real line $\mathbb{R}$ as its symmetry axis. Thus there must exist a constant $P_s > 0$ for which the unequality

$$|1-z|^{-s} \leq P_s \operatorname{Re}\Big((1-z)^{-s}\Big), \quad \operatorname{Re} z > 0$$

holds. From (60) we finally obtain

$$\log |p_n(z)| \leq C_1 \operatorname{Re}\Big((1-z)^{-s}\Big), \quad z \in \gamma.$$

The last expression is an inequality between a subharmonic function and a harmonic function which is valid on a closed curve. By the maximum principle for subharmonic functions, this inequality must also be valid on the inside of the curve, and in particular for $z = r \in (0,1)$. Hence

$$\log |p_n(r)| \leq \frac{C_1}{(1-r)^s}, \quad r \in (0,1)$$

and consequently

$$\log |\theta(r)| + \log |p_n(r)| \leq -a\frac{1+r}{1-r} + \frac{C_1}{(1-r)^s}.$$

Note that for r sufficiently close to 1, the right-hand side expression is always strictly *negative* and bounded away from 0. This is a clear contradiction to $\theta(z)p_n(z) \to 1, z \in \mathbb{D}$. Thus θ cannot be cyclic. □

6.2. **Some $\mathcal{H}(b)$-theory.** The following description of $\mathcal{H}(b)$-spaces is very convenient in connection with various functional-analytic arguments. It has been introduced in [2], and was later used in [20] and [22], to prove approximation results in classes of $\mathcal{H}(b)$-spaces. We will employ it in a similar way below. Recall that the symbol P_+ denotes the orthogonal projection operator $P_+ : \mathcal{L}^2(\mathbb{T}) \to H^2$, and $\mathcal{L}^2(E)$ denotes the subspace of those $g \in \mathcal{L}^2(\mathbb{T})$ which live only on the measurable subset $E \subset \mathbb{T}$.

Proposition 6.5. *Let b be an extreme point of the unit ball of H^∞,*

$$\Delta_b(x) = \sqrt{1-|b(x)|^2}, \quad x \in \mathbb{T}, \tag{61}$$

and E be the carrier set of Δ_b:

$$E = \{x \in \mathbb{T} : \Delta_b(x) > 0\}.$$

For $f \in \mathcal{H}(b)$ the equation

$$P_+\overline{b}f = -P_+\Delta_b g \tag{62}$$

has a unique solution $g \in \mathcal{L}^2(E)$, and the map $J : \mathcal{H}(b) \to H^2 \oplus \mathcal{L}^2(E)$ defined by

$$Jf = (f, g),$$

is an isometry. Moreover,

$$J(\mathcal{H}(b))^{\perp} = \Big\{(bh, \Delta_b h) : h \in H^2\Big\}. \tag{63}$$

Next comes a very useful corollary which is well-known and can be proved by other means (see [9], [10] for other derivations).

Corollary 6.6. *Let E and Δ_b be as in Proposition 6.5. For any $s \in \mathcal{L}^2(E)$, the function*

$$f = P_+ \Delta_b s$$

is a member of $\mathcal{H}(b)$ and, in the notation of Proposition 6.5, we have

$$Jf = (f, -\overline{b}s).$$

Moreover, if b is extreme and s is non-zero, then f is non-zero.

Proof. Indeed,

$$P_+\overline{b}f = P_+\overline{b}P_+\Delta_b s = P_+\overline{b}\Delta_b s = P_+\Delta_b\overline{b}s$$

and so (62) holds for the pair

$$(f, g) := (P_+\Delta_b s, -\overline{b}s).$$

If b is extreme then $\log \Delta_b \notin \mathcal{L}^1(\mathbb{T})$, and it follows readily that also $\log(\Delta_b|s|) \notin \mathcal{L}^1(\mathbb{T})$. Then

$$\Delta_b|s| \notin \overline{zH^2} = \ker P_+.$$

Hence $P_+\Delta_b s$ must be non-zero. □

Corollary 6.7. *The Toeplitz operator $T_{\overline{b}}$ acts boundedly on $\mathcal{H}(b)$. If $f \in \mathcal{H}(b)$ and $Jf = (f, g)$, then*

$$T_{\overline{b}}f = (T_{\overline{b}}f, \overline{b}g).$$

Proof. Again, we only need to verify that (62) holds for the given pairs. We leave out the details. □

6.3. **Main tool in the proof of Theorem D: residuals.** In order to state and use one of the main results from [23], we will now need to introduce the notion of the residual set of a weight w on $\mathbb{T}$.

Definition 6.8. (Residual sets of weights) Let $w \in \mathcal{L}^1(\mathbb{T})$ and consider the set

$$E = \{x \in \mathbb{T} : w(x) > 0\},$$

which is well-defined only up to a set of m-measure zero. We let $\operatorname{res}(w)$ ("residual" of w) be the set

$$\operatorname{res}(w) = E \setminus \operatorname{core}(w),$$

where $\operatorname{core}(w)$ is the set appearing in Definition 1.1.

Since E might only be defined up to a set of m-measure zero, the same is true for the residual $\operatorname{res}(w)$ of any weight w. However, this will never be a problem for us.

We have introduced the residuals because of their crucial role in the following special case of [23, Theorem A].

Lemma 6.9. *Assume that $w \in \mathcal{L}^1(\mathbb{T})$ is a weight for which $res(w)$ has positive m-measure. Let $w_r = w|res(w)$ be the restriction of the weight w to the set $res(w)$. Then we have the containment*

$$\mathcal{L}^2(w_r dm) \subset \mathcal{P}^2(\mu)$$

whenever μ is of the form (1) *with G satisfying* (ExpDec).

6.4. **Proof of Theorem** *D***.** In Theorem *D*, it is obvious that $(ii) \Rightarrow (i)$. Indeed, any admissible sequence satisfies (i) of Definition 5.1. We can thus prove the theorem by showing validity of the implications $(i) \Rightarrow (iii)$ and $(iii) \Rightarrow (ii)$.

Let us first show that $(i) \Rightarrow (iii)$, and so we assume that $f(z) = \sum_{n\geq 0} f_n z^n \in \mathcal{H}(b)$ is non-zero and that it satisfies

$$|f_n| \leq A \exp\Big(- cn^{1/2}\Big) \tag{64}$$

for some constants $A > 0$ and $c > 0$. We may assume that b does not vanish at any point in $\mathbb{D}$, else (iii) certainly holds. Similarly to as it was done in the proof of Proposition 6.1, we may now use (64) to produce a sequence

$$M = \{M_n\}_{n\geq 0} = \{M_n(1, c')\}_{n\geq 0},$$

and consequently a measure

$$\begin{aligned} d\mu &= d\mu_{\mathbb{D}} + d\mu_{\mathbb{T}} \\ &= \exp\Big(- \frac{c'}{(1-|z|)}\Big) dA(z) + \Delta_b \, dm, \end{aligned}$$

such that the identity map between $\mathcal{P}^2(\mu_{\mathbb{D}})$ and $H_2(M)$ is an isometry, and moreover such that $f \in H_2^*(M)$. By Proposition 5.11, the space $H_2^*(M)$ is invariant under Toeplitz operators with co-analytic symbols, and consequently we also have $T_{\bar{b}} f \in H_2^*(M)$. By

Corollary 6.7, Proposition 6.5 and in particular by formula (62), $T_{\bar{b}}f = P_+\bar{b}f \in \mathcal{H}(b) \cap H_2^*(M)$, and we have a representation

$$T_{\bar{b}}f = P_+\Delta_b g \tag{65}$$

for some $g \in \mathcal{L}^2(E)$, where Δ_b is given by (61). The kernel of the operator $T_{\bar{b}}$ is the model space $\mathcal{K}_{I_b}$, where I_b is the inner factor of b. Since b does not vanish in $\mathbb{D}$, it follows that I_b is a purely singular inner function. By Proposition 6.1 and a simple argument based on Lemma 5.10 one deduces that the space $\mathcal{K}_{I_b}$ does not contain any non-zero function in $H_2^*(M)$, so $f \notin \mathcal{K}_{I_b}$. Consequently $T_{\bar{b}}f \neq 0$, since the kernel of this Toeplitz operator is precisely $\mathcal{K}_{I_b}$. It follows also that $\Delta_b g \neq 0$. If

$$T_{\bar{b}}f(z) = \sum_{n\geq 0} c_n z^n$$

is the Taylor expansion of $T_{\bar{b}}f(z)$, then a consequence of the membership $T_{\bar{b}}f \in H_2^*(M)$ is that the function

$$F(z) := \sum_{n\geq 0} \frac{c_n}{M_n} z^n$$

is a member of $H_2(M)$. The function F lives on $\mathbb{D}$, the function g lives on $\mathbb{T}$, and hence $F - g$ defines a measurable function on $\overline{\mathbb{D}}$. Let

$$d\mu = \exp\Big(- \frac{c'}{(1-|z|)}\Big) dA(z) + \Delta_b \, dm. \tag{66}$$

The condition $F \in H_2(M)$ means simply that F is square-integrable with respect to the part $\mu_{\mathbb{D}}$ of μ which lives on $\mathbb{D}$. The containment $g \in \mathcal{L}^2(\Delta_b dm) = \mathcal{L}^2(\mu_{\mathbb{T}})$. follows from boundedness of Δ_b and the containment $g \in L^2(E)$. Thus $F - g \in \mathcal{L}^2(\mu)$. The representation (65) tells us that the positive Fourier coefficients $\{c_n\}_{n\geq 0}$ of $T_{\bar{b}}f$ and of $\Delta_b g$ coincide. Thus our definitions imply that the function $F - g$ is orthogonal to the analytic polynomials in $\mathcal{L}^2(\mu)$. Since $\Delta_b g \neq 0$, the function g is a non-zero element in $\mathcal{L}^2(\mu_{\mathbb{T}})$. The conclusion is that there exists an element (namely $F - g$) inside $\mathcal{L}^2(\mu)$ which is orthogonal to $\mathcal{P}^2(\mu)$ and which does not vanish identically on the circle $\mathbb{T}$. If there existed no interval on which $\log \Delta_b$ was integrable, then $\text{core}(w) = \varnothing$, and so Lemma 6.9 implies that the entire space $\mathcal{L}^2(\Delta_b dm)$ would be contained in $\mathcal{P}^2(\mu)$. Clearly that would be a contradiction to $F - g$ being orthogonal to $\mathcal{P}^2(\mu)$. Thus such an interval exists, and we have proved that $(i) \Rightarrow (iii)$.

The implication $(iii) \Rightarrow (ii)$ is easier. Let $M = \{M_n\}_{n\geq 0}$ be an admissible sequence. We must show that $\mathcal{H}(b)$ contains a function in $H_1^*(M)$. If b vanishes at some point of $\mathbb{D}$, then the implication is trivial. Assume therefore that $\log \Delta_b$ is integrable on some (say, open) interval I which is not all of $\mathbb{T}$, and let $w = \Delta_b | I$ be the restriction of Δ_b to the interval I.

By Proposition 5.3 and Proposition 5.9 there exists a function G which satisfies (ExpDec), (LogLogInt), with corresponding moment sequence

$$\widetilde{M} = \{\widetilde{M}_n\}_{n\geq 0}$$

satisfying

$$\widetilde{M}_n \leq M_n^2, \quad n \geq 0.$$

If

$$d\mu(z) = G(1-|z|)dA(z) + w(z)dm(z),$$

then the space $\mathcal{P}^2(\mu)$ is irreducible by Definition 1.3, since core(w) coincides with I, which is a carrier of w. By irreducibility we have that $\mathcal{L}^2(w\,dm) \not\subset \mathcal{P}^2(\mu)$. So there must exist a non-zero element $F - g \in \mathcal{L}^2(\mu)$, with F being an analytic function on $\mathbb{D}$ and g living on $I \subset \mathbb{T}$, which is orthogonal to $\mathcal{P}^2(\mu)$ in $\mathcal{L}^2(\mu)$. We can't have $g \equiv 0$, for then the Taylor coefficients of F would all vanish by the orthogonality to analytic monomials, and consequently $F - g$ would reduce to the zero element. The orthogonality means that

$$F_n\widetilde{M}_n = (wg)_n, \quad n \geq 0$$

where $\{F_n\}_{n\geq 0}$ are the Taylor coefficients of F and $(wg)_n$ are the non-negative Fourier coefficients of wg. For $n \geq 0$ we have the estimate

$$\begin{aligned} |(wg)_n|^2 &= |F_n\widetilde{M}_n|^2 \\ &\leq \widetilde{M}_n \sum_{m\geq 0} |F_m|^2 \widetilde{M}_m \\ &= \widetilde{M}_n \|F\|^2_{\mu_{\mathbb{D}}} \\ &\leq M_n^2 \|F\|^2_{\mu_{\mathbb{D}}}. \end{aligned}$$

Thus P_+wg is a member of $H_1^*(M)$. Since g lives on I, we have by Corollary 6.6 that $P_+wg = P_+\Delta_b g \in \mathcal{H}(b)$. This function is non-zero since $\log(w|g|) \notin \mathcal{L}^1(\mathbb{T})$ by the choice of $I \neq \mathbb{T}$. Thus $(iii) \Rightarrow (ii)$, and we have completed our proof of Theorem D.

7. Density in $\mathcal{H}(b)$ of functions with rapid spectral decay

The main result of [22] characterizes the density of the functions in $\mathcal{H}(b)$ which have Taylor series $f(z) = \sum_{n\geq 0} f_n z^n$ satisfying $|f_n| = \mathcal{O}(1/n^k)$, for positive k. The characterization is in terms of the structure of M_z-invariant subspaces of $\mathcal{P}^2(\mu)$ with μ of form (1) and $G(t) = t^k$, $k \geq 0$. The proofs in [22] in fact carry over more-or-less verbatim from the case considered there to many other function classes defined by their spectral size, with the family of functions defined by conditions such as (14) being no exception. Thus, in fact, Theorem E is more or less a direct consequence of Definition 1.3, Theorem A and Theorem B. However for reasons of completeness of the present work, we outline an argument which is in parts new, leads to a proof of Theorem E, but also gives additional bits of information regarding which functions in $\mathcal{H}(b)$ lie outside of the closure of functions satisfying spectral decay properties as in (RSD).

As before, we let $\Delta_b(x) = \sqrt{1 - |b(x)|}$ for $x \in \mathbb{T}$, and

$$b = BS_\nu b_0$$

be the inner-outer factorization of b, with B a Blaschke product, S_ν a singular inner function and b_0 an outer function. We denote also by $I_b = BS_\nu$ the whole inner factor of b.

The next lemma is similar to a result from [24].

Lemma 7.1. *Let $w \in \mathcal{L}^1(\mathbb{T})$ be non-negative, and assume that for some $g \in \mathcal{L}^2(w\,dm)$ the function P_+wg satisfies* (RSD)*. Then gw vanishes on res(w).*

Proof. Consider the space $\mathcal{P}^2(\mu)$ with μ constructed according to $(T1)$, and with the parameters $\beta = 1$ and $c > 0$ chosen so that if $M = \{M_n\}_{n\geq 0}$ is the sequence of moments corresponding to $\mu_{\mathbb{D}}$, then $P_+wg \in H_2^*(M)$. This is easily achieved by Proposition 5.8 and our assumption on f satisfying (RSD). Let h be some bounded measurable function living on res(w), and $\{p_n\}_{n\geq 0}$ be a sequence of analytic polynomials which converges to h in the norm of $\mathcal{P}^2(\mu)$. This is possible by Lemma 6.9. In particular, this convergence implies that $p_n \to 0$ in $\mathcal{P}^2(\mu_{\mathbb{D}})$, or in other words, $p_n \to 0$ in $H_2(M)$. The computation

$$\begin{aligned} 0 &= \langle 0, P_+wg \rangle \\ &= \lim_{n\to\infty} \langle p_n, P_+wg \rangle \\ &= \lim_{n\to\infty} \langle p_n, wg \rangle \\ &= \int_{\mathbb{T}} h\overline{g}\, w dm. \end{aligned}$$

is thus valid, where the pairing $\langle \cdot, \cdot \rangle$ denotes the Cauchy duality from (53). Since h is an arbitrary bounded measurable function living on res(w), it follows that $gw \equiv 0$ on res(w). □

Proposition 7.2. *Assume that the set res(Δ_b) has positive m-measure, and let $s \in \mathcal{L}^2(\mathbb{T})$ be a non-zero function which vanishes outside of the set res(Δ_b). Then the non-zero function*

$$f = P_+\Delta_b s \in \mathcal{H}(b)$$

lies outside of the norm-closure in $\mathcal{H}(b)$ *of functions satisfying* (RSD) *for some (and any) fixed* $c > 0$.

Proof. Seeking a contradiction, assume that $\{h_n\}_n$ is a sequence of functions in $\mathcal{H}(b)$ which satisfy (RSD) for some $c > 0$ and which converge in the norm of $\mathcal{H}(b)$ to the given f. In the notation of Proposition 6.5, we consider $Jh_n = (h_n, k_n) \in H^2 \oplus \mathcal{L}^2(E)$ and $Jf = (f, g) \in H^2 \oplus \mathcal{L}^2(E)$, where $g = -\overline{b}s$ according to Corollary 6.6. By Corollary 6.7, $T_{\overline{b}}h_n$ converges to $T_{\overline{b}}f$ in the norm of $\mathcal{H}(b)$, and since the embedding J of Proposition 6.5 is an isometry, Corollary 6.7 moreover implies that $\overline{b}k_n$ converges to $\overline{b}g = -\overline{b^2}s$ in $\mathcal{L}^2(\mathbb{T})$. In particular, this implies that k_n cannot all simultaneously vanish on $\text{res}(\Delta_b)$, since s lives only on that set. But $T_{\overline{b}}h_n$ satisfies (RSD) (since h_n does), and $T_{\overline{b}}h_n = P_+\overline{b}h_n = P_+\Delta_b\overline{b}k_n$ by Corollary 6.7. Thus by Lemma 7.1, the functions $\Delta_b\overline{b}k_n$ must vanish on $\text{res}(\Delta_b)$, and consequently k_n must vanish on $\text{res}(\Delta_b)$, since $\overline{b}\Delta_b$ is non-zero m-almost everywhere on that set. This is the desired contradiction. □

We have now proved that it is necessary for $\text{core}(\Delta_b)$ to be a carrier for Δ_b if functions satisfying (RSD) are to be dense in $\mathcal{H}(b)$. In the next proposition, we assume that $\text{core}(\Delta_b)$ is a carrier for Δ_b, and show that if S_ν is the singular inner factor of b and the singular measure ν places some portion of its mass outside of the core of Δ_b, then again functions satisfying (RSD) are not dense in $\mathcal{H}(b)$. And again, we do it by exhibiting explicit functions in $\mathcal{H}(b)$ which cannot be approximated in this way.

Proposition 7.3. *Assume that core*(Δ_b) *is a carrier for* Δ_b *and that*

$$\nu\big(\mathbb{T} \setminus \mathit{core}(\Delta_b)\big) > 0,$$

where S_ν *is the singular inner factor of* b. *Decompose the measure* ν *as*

$$\nu = \nu_r + \nu_c,$$

where ν_r *is the restriction of* ν *to the set* $\mathbb{T} \setminus \mathit{core}(\Delta_b)$, *and* ν_c *is the restriction of* ν *to the set core*(Δ_b). *Then all functions in the subspace*

$$(b/S_{\nu_r})\mathcal{K}_{S_{\nu_r}} = BS_{\nu_c}b_0\mathcal{K}_{S_{\nu_r}} \subset \mathcal{H}(b)$$

are orthogonal in $\mathcal{H}(b)$ *to functions satisfying* (RSD), $\mathcal{K}_{S_{\nu_r}}$ *being the model space generated by the singular inner function* S_{ν_r}.

Proof. Take a function $f = BS_{\nu_c}b_0s$, where $s \in \mathcal{K}_{S_{\nu_r}}$, and h satisfying (RSD). In the notation of Proposition 6.5, a computation shows that $Jf = (f, g)$, where

$$g = \Delta_b\overline{S_{\nu_r}}s.$$

Let $\mathcal{P}^2(\mu)$ and $H_2(M) = \mathcal{P}^2(\mu_{\mathbb{D}})$ be as in the proof of Lemma 7.1, with $w = \Delta_b$ and the sequence M being chosen so that $h \in H_2^*(M)$. This time the space $\mathcal{P}^2(\mu)$ is irreducible, and by Theorem A the singular inner function S_{ν_r} is cyclic in $\mathcal{P}^2(\mu)$. Hence there exists a sequence of analytic polynomials $\{p_n\}_n$ such that $S_{\nu_r} p_n$ converges to the function $s \in H^2$ in the norm of $\mathcal{P}^2(\mu)$, and in particular in the norm of $H_2(M)$. Multiplying this sequence by $BS_{\nu_c} b_0$, it follows that bp_n converges to f in $H_2(M)$. Simultaneously, the $\mathcal{P}^2(\mu)$-convergence implies that $S_{\nu_r} p_n$ converges to s in $\mathcal{L}^2(\Delta_b dm)$, and since S_{ν_r} is unimodular on $\mathbb{T}$, in fact we have that p_n converge to $\overline{S_{\nu_r}} s$ in $\mathcal{L}^2(\Delta_b dm)$. Let $Jh = (h, k)$ be the corresponding pair for h. If $\langle \cdot, \cdot \rangle$ is the duality pairing in (53) and $\langle \cdot, \cdot \rangle_{\mathcal{L}^2}$ is the usual $\mathcal{L}^2$-pairing of square-integrable functions on $\mathbb{T}$, then we can use the above claims, and the fact that $h \in H_2^*(M)$, to compute

$$\begin{aligned}
\langle h, f \rangle + \langle k, g \rangle_{\mathcal{L}^2} &= \langle h, f \rangle + \langle k, \Delta_b \overline{S_{\nu_r}} s \rangle_{\mathcal{L}^2} \\
&= \lim_{n\to\infty} \langle h, bp_n \rangle + \langle k, \Delta_b p_n \rangle_{\mathcal{L}^2} \\
&= \lim_{n\to\infty} \langle h, bp_n \rangle_{\mathcal{L}^2} + \langle k, \Delta_b p_n \rangle_{\mathcal{L}^2} \\
&= \lim_{n\to\infty} \langle P_+(\bar{b}h + \Delta_b k), p_n \rangle_{\mathcal{L}^2} \\
&= \lim_{n\to\infty} \langle 0, p_n \rangle_{\mathcal{L}^2} = 0.
\end{aligned}$$

In the last step we used condition (62) for the pair (h, k). Since the embedding J in Proposition 6.5 is an isometry, it follows that f is orthogonal to h in $\mathcal{H}(b)$. □

Proof of Theorem E. We see from Proposition 7.2 and Proposition 7.3 above that condition (iii) in Theorem E is necessary in order for (i) to hold. Since (ii) implies (i), it suffices thus to show that (iii) implies (ii). The argument is essentially same as the one appearing in [22] and [20], we include it only for completeness.

Just as in the proof of Theorem D, given an admissible sequence $M = \{M_n\}_{n\geq 0}$ we use Proposition 5.3 and Proposition 5.9 to obtain G satisfying (ExpDec), (LogLogInt), with moment sequence $\widetilde{M} = \{\widetilde{M}_n\}_{n\geq 0}$ satisfying $\widetilde{M}_n \leq M_n^2$ for $n \geq 0$. We must show that $\mathcal{H}(b) \cap H_1^*(M)$ is dense in $\mathcal{H}(b)$. By Lemma 5.10 it will suffice to show that $H_2^*(M^2) \cap \mathcal{H}(b)$ is dense in $\mathcal{H}(b)$.

The space $\mathcal{P}^2(\mu)$ constructed from the measure

$$d\mu(z) = G(1 - |z|)dA(z) + \Delta_b^2(z)dm(z)$$

is irreducible by Definition 1.3. Let us assume that $f \in \mathcal{H}(b)$ is orthogonal to $H_2^*(M^2) \cap \mathcal{H}(b)$. We will show that $f = 0$, which will prove Theorem D. Because the mapping J in

Proposition 6.5 is an isometry, it follows that Jf is orthogonal to $J(H_2^*(M^2) \cap \mathcal{H}(b))$. Note that $J(H_2^*(M^2) \cap \mathcal{H}(b))$ is a subset of $H_2^*(M^2) \oplus \mathcal{L}^2(E)$, and under the duality pairing (53) between $H_2(M^2)$ and $H_2^*(M^2)$, we have

$$J(H_2^*(M^2) \cap \mathcal{H}(b)) = \cap_{h \in H^2} \ker l_h, \tag{67}$$

where l_h is the functional on $H_2^*(M^2) \oplus \mathcal{L}^2(E)$ which acts by the formula

$$l_h(f,g) := \langle f, bh \rangle + \langle g, \Delta_b h \rangle_{\mathcal{L}^2}.$$

This follows readily from Proposition 6.5 (see, for instance, the argument in [22]). The fact that Jf annihilates $J(H_2^*(M^2) \cap \mathcal{H}(b))$ and (67) holds implies that Jf is contained in the weak-star closure of the linear manifold $\{l_h\}_{H^2} \subseteq H_2(M^2) \oplus \mathcal{L}^2(E)$. Since the pairing between $H_2(M)$ and $H_2^*(M^2)$ is reflexive and $\{l_h\}_{h \in H^2}$ is a convex set, basic functional analysis says that, in fact, Jf is contained in the norm-closure of $\{l_h\}_{h \in H^2}$. Thus there exists a sequence $\{h_k\}_{k \geq 1}$ with $h_k \in H^2$ such that

$$(bh_k, \Delta_b h_k) \to Jf := (f, g) \tag{68}$$

in the norm of $H_2(M) \oplus \mathcal{L}^2(E)$. Multiply the second coordinate by b to obtain

$$(bh_k, \Delta_b bh_k) \to Jf := (f, bg). \tag{69}$$

But the inequalities $\widetilde{M_n} \leq M_n^2$ imply that bh_k converges to f also in the space $\mathcal{P}^2(\mu_{\mathbb{D}})$, and so in fact (69) tells us that $\{bh_k\}_n$ is a Cauchy sequence in the irreducible space $\mathcal{P}^2(\mu)$, to which Theorem B applies. If I_b is the inner factor of b, then Theorem B implies that $f/I_b \in H^2$, and by the irreducibility of $\mathcal{P}^2(\mu)$ the sequence bh_k on $\mathbb{T}$ must converge to boundary function of f on $\mathbb{T}$. Thus $g = \Delta_b f/b$. All in all, $Jf = (f, \Delta_b f/b)$. By Proposition 6.5 we get that

$$0 = P_+(\bar{b}f + \Delta_b g) = P_+(\bar{b}f + \Delta_b^2 f/b) = P_+(|b|^2 f/b + \Delta_b^2 f/b) = P_+(f/b). \tag{70}$$

From the above computation we infer that, in terms of boundary values, we have $f/b = \bar{b}f + \Delta_b g \in \mathcal{L}^2(\mathbb{T})$, and consequently f/b has square-integrable boundary values. Since $f/I_b \in H^2$, it follows from the classical Smirnov maximum principle that $f/b \in H^2$. Then f/b is an analytic function which projects to 0 under P_+, which implies that $f/b = 0$, and consequently $f = 0$. □

8. Proof of Theorem C

A proof of Theorem C relies on a judicious application of Lemma 7.1.

Proof of Theorem C. If C_ν satisfies (RSD), then the function $f(z) = \sum_{n\geq 0} \nu_n z^n$, $z \in \mathbb{T}$, is certainly smooth on $\mathbb{T}$ and it has an analytic extension to $\mathbb{D}$. Since the Cauchy transform of the measure $d\nu - f \cdot dm$ vanishes in $\mathbb{D}$, this measure must be absolutely continuous with respect to m by the classical theorem of brothers Riesz. Hence $d\nu$ is also absolutely continuous. Let $g \in \mathcal{L}^1(\mathbb{T})$ be its Radon-Nikodym derivative, so that $d\nu = g \cdot dm$. Set $f = \mathcal{C}_\nu = \mathcal{C}_g$, which by our assumption is a function satisfying (RSD). Unfortunately, we cannot directly apply Lemma 7.1 since we do not necessarily have that $g \in \mathcal{L}^2(\mathbb{T})$. We must take care of this slight inconvenience to prove the theorem.

It is not hard to see from Lemma 5.10 and Proposition 5.11 that $T_{\overline{h}} f$ also satisfies (RSD), where $T_{\overline{h}}$ is any co-analytic Toeplitz operator with bounded symbol $h \in H^\infty$. Moreover, $T_{\overline{h}} f$ has the representation

$$T_{\overline{h}} f(z) = \mathcal{C}_{\overline{h}g}(z), \quad z \in \mathbb{D}.$$

The above formula can be derived by first showing by simple algebraic manipulations that it holds for analytic monomials, and then for analytic polynomials. Finally, fix a uniformly bounded sequence of polynomials $\{p_n\}_{n\geq 1}$ which converges to h pointwise m-almost everywhere on $\mathbb{T}$ (the polynomials p_n could be taken to be the Cesàro means of the partial sums of the Taylor series of h). For such a sequence we readily see from the dominated convergence theorem that

$$T_{\overline{h}} f(z) = \lim_{n\to\infty} T_{\overline{p_n}} f(z) = \lim_{n\to\infty} \mathcal{C}_{\overline{p_n}g}(z) = \mathcal{C}_{\overline{h}g}(z), \quad z \in \mathbb{D}.$$

Since $g \in L^1(\mathbb{T})$, in particular we have that $\log^+ |g| \in L^1(\mathbb{T})$, and this means that an outer function $h \in H^\infty$ exists which satisfies the boundary value equation

$$|h(x)| = \min\Big(1, 1/|g(x)|\Big)$$

for m-almost every $x \in \mathbb{T}$. Set also

$$w(x) = \min\Big(1, |g(x)|\Big).$$

Now, we can write

$$\overline{h}g = \overline{h}\frac{g}{w}w := u \cdot w$$

with

$$u := \overline{h}\frac{g}{w}.$$

It is easily checked that u is a bounded measurable function satisfying $|u(x)| = 1$ for m-almost every $x \in \mathbb{T}$. Then

$$T_{\overline{h}} f(z) = \mathcal{C}_{\overline{h}g} = P_+ uw, \quad z \in \mathbb{D}$$

and therefore Lemma 7.1 can be applied to conclude that uw vanishes on $\operatorname{res}(w)$. Since u is modular, w vanishes on $\operatorname{res}(w)$, and consequently the set

$$\{x \in \mathbb{T} : w(x) > 0\} = \{x \in \mathbb{T} : |g(x)| > 0\}$$

coincides with $\operatorname{core}(w)$, up to a set of m-measure zero. For any interval I contained in $\operatorname{core}(w)$ it follows from the pointwise inequality $|g| \geq w$ and the definition of $\operatorname{core}(w)$ that

$$\int_I \log |g| \, dm \geq \int_I \log w \, dm > -\infty.$$

Thus g has structure as claimed in the statement of Theorem C, and the proof is complete. □

References

[1] J. Agler and J. E. McCarthy. *Pick Interpolation and Hilbert Function Spaces*, volume 44 of *Graduate Studies in Mathematics*. American Mathematical Society, 2002.

[2] A. Aleman and B. Malman. Density of disk algebra functions in de Branges–Rovnyak spaces. *C. R. Math. Acad. Sci. Paris*, 355(8):871–875, 2017.

[3] A. Beurling. A critical topology in harmonic analysis on semigroups. *Acta Mathematica*, 112(1):215–228, 1964.

[4] A. Beurling. Analytic continuation across a linear boundary. *Acta Mathematica*, 128(1):153–182, 1972.

[5] J. Cima, A. Matheson, and W. Ross. *The Cauchy transform*, volume 125 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2006.

[6] J. B. Conway. *The theory of subnormal operators*. American Mathematical Soc., 1991.

[7] B. M. Davis and J. E. McCarthy. Multipliers of de Branges spaces. *Michigan Math. J*, 38(2):225–240, 1991.

[8] O. El-Fallah, K. Kellay, and K. Seip. Cyclicity of singular inner functions from the corona theorem. *Journal of the Institute of Mathematics of Jussieu*, 11(4):815–824, 2012.

[9] E. Fricain and J. Mashreghi. *The theory of $\mathcal{H}(b)$ spaces. Vol. 1*, volume 20 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2016.

[10] E. Fricain and J. Mashreghi. *The theory of $\mathcal{H}(b)$ spaces. Vol. 2*, volume 21 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2016.

[11] J. Garnett. *Bounded analytic functions*, volume 236. Springer Science & Business Media, 2007.

[12] J. Garnett and D. Marshall. *Harmonic measure*, volume 2. Cambridge University Press, 2005.

[13] V. P. Havin and B. Jöricke. *The uncertainty principle in harmonic analysis*, volume 72 of *Encyclopaedia Math. Sci.* Springer, Berlin, 1995.

[14] S. V. Khrushchev. The problem of simultaneous approximation and of removal of the singularities of Cauchy type integrals. *Trudy Matematicheskogo Instituta imeni VA Steklova*, 130:124–195, 1978.

[15] P. Koosis. *Introduction to H_p spaces*, volume 115 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, second edition, 1998.

[16] B. Korenblum. An extension of the Nevanlinna theory. *Acta Mathematica*, 135:187–219, 1975.

[17] B. Korenblum. A Beurling-type theorem. *Acta Mathematica*, 138(1):265–293, 1977.

[18] T. L. Kriete and B. D. MacCluer. Mean-square approximation by polynomials on the unit disk. *Transactions of the American Mathematical Society*, 322(1):1–34, 1990.

[19] A. Limani and B. Malman. An abstract approach to approximation in spaces of pseudocontinuable functions. *Proceedings of the American Mathematical Society*, 150(06):2509–2519, 2022.

[20] A. Limani and B. Malman. Constructions of some families of smooth Cauchy transforms. *Canadian Journal of Mathematics*, 2023.

[21] A. Limani and B. Malman. On model spaces and density of functions smooth on the boundary. *Revista Matemática Iberoamericana*, 39(3):1059–1071, 2023.

[22] A. Limani and B. Malman. On the problem of smooth approximations in $\mathcal{H}(b)$ and connections to subnormal operators. *Journal of Functional Analysis*, 284(5):109803, 2023.

[23] B. Malman. Revisiting mean-square approximation by polynomials in the unit disk. *arXiv preprint arXiv:2304.01400*, 2023.

[24] B. Malman. Thomson decompositions of measures in the disk. *to appear in Transactions of the American Mathematical Society*, 2023.

[25] T. Ransford. *Potential theory in the complex plane*. Number 28 in London Mathematical Society Student Texts. Cambridge university press, 1995.

[26] J. Roberts. Cyclic inner functions in the Bergman spaces and weak outer functions in H^p, $0 < p < 1$. *Illinois Journal of Mathematics*, 29(1):25–38, 1985.

[27] D. Sarason. *Sub-Hardy Hilbert spaces in the unit disk*, volume 10 of *University of Arkansas Lecture Notes in the Mathematical Sciences*. John Wiley & Sons, Inc., New York, 1994.

[28] B. Sz.-Nagy, C. Foias, H. Bercovici, and L Kérchy. *Harmonic analysis of operators on Hilbert space*. Universitext. Springer, New York, second edition, 2010.

[29] A. L Vol'berg and B. Ërikke. Summability of the logarithm of an almost analytic function and a generalization of the Levinson-Cartwright theorem. *Mathematics of the USSR-Sbornik*, 58(2):337, 1987.

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