A PROOF OF THE WEIGHTED KORENBLUM-ROBERTS THEOREM BASED ON A NEW ROBERTS-TYPE DECOMPOSITION

BARTOSZ MALMAN

Abstract. We provide an inhomogeneous Roberts-type decomposition for singular Borel measures adapted to the notion of so-called w-associated Beurling-Carleson sets and Hausdorff functionals from [Mal26]. As opposed to the classical decomposition into countably many "good" layers, the new decomposition has only two parts: a single "good" layer with local mass distribution controlled by a gauge function h and a non-negative function w, and a residual layer with controlled total mass. Using this new decomposition, we give a much shorter proof of the weighted version of the Korenblum-Roberts Theorem on cyclic singular inner functions in certain inhomogeneous weighted Lebesgue spaces from [Mal26].

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1. Introduction and main result

1.1. **Gauge functions and sets associated to a weight.** Let h be a gauge function. By this we mean that $h(x)$ is a non-negative continuous increasing function defined for $x \in [0,1]$, and $h(0) = 0$. We additionally suppose that $\lim_{x\to 0+} h(x)/x = +\infty$ and that $x \mapsto h(x)/x$ is decreasing. Two principal examples which we consider here are $h(x) = x\log(e/x)$ and $h(x) = x^\alpha$, $\alpha \in (0,1)$.

Given a gauge function h we define the class of h-Beurling-Carleson sets $\mathcal{BC}_h$ to consist of compacts $E \subset \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ for which the system $\mathrm{comp}(E)$ of maximal open complementary arcs constituting $\mathbb{T} \setminus E$ satisfies

$$\sum_{\ell \in \mathrm{comp}(E)} h(|\ell|) < \infty.$$

A portion of mathematical results presented in this note have been obtained with the aid of ChatGPT 5.6 Pro. See the Methodology Section 1.4.

The subfamily of $\mathcal{BC}_h$ consisting of sets of Lebesgue measure zero is denoted by $\mathcal{BC}_h^0$.

Let m denote the normalized Lebesgue measure on $\mathbb{T}$ ($m(\mathbb{T}) = 1$). Given a non-negative weight w on $\mathbb{T}$, namely a Borel function $w \in L^1(\mathbb{T})$, we define the family of w-associated $\mathcal{BC}_h$-sets as

$$\mathrm{Assoc}\mathcal{BC}_h(w) := \{E \in \mathcal{BC}_h : \int_E \log w \, dm > -\infty.\}$$

We note the extreme cases $\mathrm{Assoc}\mathcal{BC}_h(0) = \mathcal{BC}_h^0$ if $w = 0$ vanishes completely, and $\mathrm{Assoc}\mathcal{BC}_h(w) = \mathcal{BC}_h$ if w is bounded away from zero on $\mathbb{T}$. In general, we have

$$\mathcal{BC}_h^0 \subset \mathrm{Assoc}\mathcal{BC}_h(w) \subset \mathcal{BC}_h.$$

The associated sets have been introduced in [Mal26] and appear implicitly in [BM24].

We have the set equality $\mathrm{Assoc}\mathcal{BC}_h(w) = \mathrm{Assoc}\mathcal{BC}_h(\min(w, 1))$, and in the rest of the note we shall therefore assume that $0 \le w(z) \le 1$ for $z \in \mathbb{T}$. It may also be worth to keep in mind that the family of associated sets is invariant under replacing h by a positive scalar multiple of itself, and upon replacing w by any of its powers w^c, $c > 0$.

1.2. **Weighted Korenblum-Roberts Theorem.** Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk, $H^\infty(\mathbb{D})$ be the algebra of bounded holomorphic functions on $\mathbb{D}$, and $\|\cdot\|_{\infty,\mathbb{T}}$ be the (essential) supremum norm on $\mathbb{T}$. For $g \in H^\infty(\mathbb{D})$, g on $\mathbb{T}$ is defined as usual by its ordinary non-tangential boundary values. In [Mal26] a version of the following theorem has been proved.

Theorem A. *Let $S_\nu : \mathbb{D} \to \mathbb{D}$ be a singular inner function associated to the singular finite Borel measure ν on $\mathbb{T}$, h be a gauge function and w be a measurable function satisfying $0 \le w(z) \le 1$ for $z \in \mathbb{T}$. If ν vanishes on all sets in $Assoc\mathcal{BC}_h(w)$, i.e.,*

$$\nu(E) = 0, \quad E \in Assoc\mathcal{BC}_h(w),$$

then there exists a sequence $(g_n)_n$ of functions in $H^\infty(\mathbb{D})$ such that

$$S_\nu(z) g_n(z) \to 1, \quad z \in \mathbb{D},$$

$$\sup_n |S_\nu(z) g_n(z)| \le \exp\Big(\frac{h(1-|z|)}{1-|z|}\Big), \quad z \in \mathbb{D},$$

and

$$\sup_n \|S_\nu g_n w\|_{\infty,\mathbb{T}} = \sup_n \|g_n w\|_{\infty,\mathbb{T}} \le 1.$$

In the classical setting $w = 0$ and $h(x) = x \log(e/x)$ the above result gives the sufficiency part of the Korenblum-Roberts Theorem from [Kor81] and [Rob85], which characterizes cyclic[1] singular inner function in the Bergman spaces of the unit disk as those for which the corresponding measure ν vanishes on all Beurling-Carleson sets of Lebesgue measure zero, i.e., all sets in $\mathcal{BC}_h^0$ for $h(x) = x \log(e/x)$. Necessity of the vanishing condition is comparatively much easier to establish, see Roberts paper [Rob85] and [Mal26, Section 4]. The extended formulation presented here establishes cyclicity of the corresponding singular inner functions in the larger class of $\mathcal{P}^t(\mu)$-spaces with measures of the form $d\mu(z) = dA(z) + w(z)dm(z)$, here dA being the area measure[2]. It is the vanishing of ν on the larger class $\mathrm{Assoc}\mathcal{BC}_h(w)$, compared to $\mathcal{BC}_h^0$, that allows us to ensure the sup-norm bound $\|S_\nu g_n w\|_{\infty,\mathbb{T}} \le 1$ in connection to the pointwise convergence and pointwise growth estimates appearing in Theorem A and so prove cyclicity of S_ν in the inhomogeneous spaces $\mathcal{P}^2(\mu)$ of the indicated form.

[1] By "cyclic" we always mean cyclic for the shift operator $f(z) \mapsto zf(z)$.

[2] The space $\mathcal{P}^t(\mu)$ is the closure of analytic polynomials in the Lebesgue space $L^t(\mu)$. In [Mal26] a characterization exists of the measures of the form $d\mu(z) = dA(z) + w(z)dm(z)$ for which $\mathcal{P}^t(\mu)$ itself is a space of analytic functions on $\mathbb{D}$.

The result has an important application to approximation theory in de Branges-Rovnyak spaces $\mathcal{H}(b)$, which was one of the motivations of the author for establishing it. See [LM23] for the connections to $\mathcal{H}(b)$-theory, and [Mal26] for more details on the role of Theorem A in the theory of $\mathcal{P}^t(\mu)$-spaces and the role of associated sets in problems of polynomial approximations in the plane.

The proof of Theorem A which is given in [Mal26] is long and convoluted. It relies on Korenblum's notion of premeasures, on his linear programming ideas from [Kor75], [Kor77] and on optimization theory from [BM24]. The given proof is for various reasons hard to generalize to other situations, which are considered for instance in [Ivr19] and [IN24]. Here we shall give a much shorter proof which is based on [BM24] and on a new Roberts-type decomposition for measures ν vanishing on w-associated sets. This new decomposition is hopefully of independent interest, as appears strikingly simpler than the classical one.

1.3. **Main new tool in the new proof: inhomogeneous Roberts-type decomposition.** We recall first the principle behind the classical Roberts decomposition from [Rob85].

Let ν be a finite (singular) non-negative Borel measure on the unit circle $\mathbb{T}$. According to Roberts in [Rob85], if $\nu(E) = 0$ for all $E \in \mathcal{BC}_h^0$, then the measure ν can be decomposed into countably many pieces

$$\nu = \sum_{j \geq 1} \nu_j$$

in which each piece ν_j satisfies the estimate $\nu_j(I) \leq h(|I|)$ for all intervals I belonging to a certain partition $\mathcal{P}_j$ of the circle [3]. This result enabled Roberts to give in [Rob85] a new proof of cyclicity in Bergman spaces of the singular inner functions S_ν associated to measures ν vanishing on sets in $\mathcal{BC}_h^0$. The decomposition appears also in proofs of new cyclicity theorems in work of Limani [Lim24]. In recent years, completely new applications of Roberts decomposition have appeared. Notably, they are used in work of Ivrii [Ivr19] on critical structures of inner functions, and in the work of Ivrii and Nicolau [IN24] on non-linear partial differential equations.

In this note, we will establish a Roberts-type decomposition which is in one way simpler and in one way more complicated than the original. The complication arises from replacing the mass distribution estimates defined in terms of the gauge h by a different, inhomogeneous quantity defined in terms of h and a non-negative measurable function $R : \mathbb{T} \to [0, \infty]$, namely

$$M_h(R) := \inf_{\{I_j\}_j} \left(\sum_j h(|I_j|) + \int_{\mathbb{T} \setminus \cup_j I_j} R \, dm \right). \tag{1.1}$$

Here the infimum is taken over countable families $\{I_j\}_j$ of disjoint intervals I_j. The functional has been introduced in [BM24] and used to resolve a long-standing conjecture of Kriete and MacCluer in polynomial approximation theory from [KM90]. It was also one of the main tools used to establish a version of our present Theorem A in [Mal26]. Its importance comes from its association to a certain L^1-optimization problem which we present in Theorem 3.1 and which we will use in the proof of Theorem A below.

Let us explain the expression on the right-hand side in (1.1). If $\mathbb{1}_A$ denotes the indicator function of a subset A, then $R = +\infty \cdot \mathbb{1}_A$ is the function taking the value $+\infty$ on the set $A \subset \mathbb{T}$, and vanishes elsewhere. In this case $M_h(R)$ is similar to the definition of the Hausdorff h-content of the set A, for finiteness of the expression associated to a given family $\{I_j\}_j$ in $M_h(R)$ requires $\{I_j\}_j$ to cover A (up to an error of Lebesgue measure zero, in this definition). It is for this reason we call the mapping $R \mapsto M_h(R)$ for the Hausdorff functional. In our functional version, the

[3] Roberts stated his result in terms of the gauge $h(x) = x \log(e/x)$, but his technique can be readily generalized.

family $\{I_j\}_j$ need not cover the support of R. Instead, any uncovered region has an associated cost $\int_{\mathbb{T}-(\cup_j I_j)} R\,dm$ to it.

The main new result of this note is the following Roberts-type decomposition involving the functional $M_h(R)$.

Theorem B. *Assume that ν is a (singular) finite non-negative Borel measure on $\mathbb{T}$ which vanishes on all $\mathcal{BC}_h$-sets associated to a given weight w:*

$$\nu(E) = 0, \quad E \in Assoc\mathcal{BC}_h(w).$$

Set $R(z) = \max\{\log(1/w(z)), 0\}$, $z \in \mathbb{T}$. Then, given any $\epsilon > 0$ and $\delta > 0$, there exists a partition $\mathcal{P}$ of $\mathbb{T}$ consisting of (half-open) intervals of lengths at most δ, and a decomposition

$$\nu = \nu_g + \nu_r$$

such that

$$\nu_g(I) \leq M_h(R \cdot \mathbb{1}_I), \quad I \in \mathcal{P}$$

and

$$\nu_r(\mathbb{T}) \leq \epsilon.$$

The theorem statement and its proof do not require the use of singularity of the measure ν, but the result is interesting only in that case. In fact, for singular ν, the existence of the decompositions appearing in Theorem B is equivalent to vanishing of ν on sets in Assoc$\mathcal{BC}_h(w)$. See Theorem D below, which establishes a converse statement to Theorem B.

Let us note how the new decomposition translates into the classical setting $w \equiv 0$. Then $R = +\infty$ on $\mathbb{T}$, and Assoc$\mathcal{BC}_h(w) = \mathcal{BC}_h^0$, as noted earlier. One also easily verifies that $M_h(R \cdot \mathbb{1}_I) = h(|I|)$. In this setting, the above theorem gives us a one-layer Roberts-type decomposition in which the "good" piece ν_g satisfies a homogeneous mass distribution inequality $\nu_g(I) \leq h(|I|)$ for elements I of a partition $\mathcal{P}$ of $\mathbb{T}$ into intervals of controlled but not uniform length. The "bad" residual piece ν_r has controlled total mass. In contrast, in the classical decomposition of Roberts the number of layers ν_j is infinite, but the corresponding partitions $\mathcal{P}_j$ consist of intervals of uniform length, and the residual piece ν_r does not appear. The precise translation of the new inhomogeneous result into the classical homogeneous setting reads as follows.

Corollary C. *Assume that ν is a (singular) finite non-negative Borel measure on $\mathbb{T}$ which vanishes on all $\mathcal{BC}_h^0$ sets:*

$$\nu(E) = 0, \quad E \in \mathcal{BC}_h^0.$$

Then, given any $\epsilon > 0$ and $\delta > 0$, there exists a partition $\mathcal{P}$ of $\mathbb{T}$ consisting of intervals of lengths at most δ, and a decomposition

$$\nu = \nu_g + \nu_r$$

such that

$$\nu_g(I) \leq h(|I|), \quad I \in \mathcal{P}$$

and

$$\nu_r(\mathbb{T}) \leq \epsilon.$$

Our proof of the weighted Korenblum-Roberts Theorem stated in Theorem A will follow from Theorem B by employing a standard technique of weakly approximating the singular measure ν by absolutely continuous non-negative measures $f dm$, where the bounded function f has a sufficiently "smooth" mass distribution. The existence of such a mass distribution follows from Theorem A and a duality principle from [BM24]. The proof is given in the final Section 3.

Finally, we discuss the mentioned converse result to Theorem B. Before stating the theorem, let us note that the existence of the decompositions appearing in Theorem B does not imply vanishing

on Assoc$\mathcal{BC}_h(w)$ of the absolutely continuous part of ν. Indeed, let $w(z) = 1/e$ be constant on $\mathbb{T}$, so that $R = \max\{\log(e), 0\} = 1$. Then Assoc$\mathcal{BC}_h(w) = \mathcal{BC}_h$ and setting $d\nu = dm$ to be the (normalized) Lebesgue measure on $\mathbb{T}$, and one can deduce that if the interval I is sufficiently short, then

$$M_h(R \cdot \mathbb{1}_I) = |I|.$$

Hence ν is decomposable as in Theorem B in the trivial way with $\nu = \nu_g$, but obviously $\nu(E) = |E|$ so ν does not vanish on Assoc$\mathcal{BC}_h(w) = \mathcal{BC}_h$. For this reason a statement converse to Theorem B can only concern the singular part of ν.

Theorem D. *Let w be a weight and set $R(z) = \max\{\log(1/w(z)), 0\}$. Let ν be a finite non-negative Borel measure on $\mathbb{T}$. Assume that given any $\epsilon > 0$ and $\delta > 0$, there exists a partition $\mathcal{P}$ of $\mathbb{T}$ consisting of (half-open) intervals of lengths at most δ, and a decomposition*

$$\nu = \nu_g + \nu_r$$

such that

$$\nu_g(I) \leq M_h(R \cdot \mathbb{1}_I), \quad I \in \mathcal{P}$$

and

$$\nu_r(\mathbb{T}) \leq \epsilon.$$

Then the singular part ν_s of ν vanishes on all $\mathcal{BC}_h$-sets associated to w:

$$\nu_s(E) = 0, \quad E \in Assoc\mathcal{BC}_h(w).$$

This result may be compared with [IN24, Corollary 3.4] which is a result in similar vein for the classical Roberts decomposition.

1.4. **Methodology and the use of Large Language Models.** This note is inspired in a significant part by output from the Large Language Model (LLM) ChatGPT 5.6 Pro.

The work was initiated by a prompt asking the LLM for ideas on reformulating the original homogeneous Roberts decomposition theorem (associated to a gauge function h) into one involving only one good layer. The result of the search was in principle a statement approximating the decomposition appearing in Corollary C above, together with a valid proof. The critical idea of the proof supplied by the LLM was the identification of the set B_δ appearing in Lemma 2.1 below as a $\mathcal{BC}_h^0$-set (where M_h is replaced by h in the definition of B_δ). The adaptation of this argument to the setting of the functional M_h, which appears below in Lemma 2.1, is due to the author, as are the remaining proofs in the note.

ChatGPT 5.6 Pro has also been used used to check spelling, equations, and grammatical structure, and to verify the proofs.

2. Proof of Theorem B and Theorem D

2.1. **Central notion: an embedding lemma.** The proof of Theorem B is based on the following assertion that a certain set B_δ of points "bad" for ν can be embedded in a set associated to w. In the statement below, the finiteness of the integral $\int_E R\, dm$ in the case $R = \max\{\log(1/w), 0\}$ simply means that $E \in \text{Assoc}\mathcal{BC}_h(w)$.

Lemma 2.1. *Let ν be a (singular) non-negative atomless finite Borel measure on $\mathbb{T}$, $R : \mathbb{T} \to [0, \infty]$ be a Borel measurable function, and let $\delta > 0$. Consider the set $B_\delta \subset \mathbb{T}$ consisting of all $z \in \mathbb{T}$ such that for any open arc $I \subset \mathbb{T}$ satisfying*

$$z \in I, |I| < \delta$$

we have

$$\nu(I) > M_h(R \cdot \mathbb{1}_I).$$

There exists a set $E \in \mathcal{BC}_h$ which contains B_δ such that $\int_E R\,dm < \infty$.

Proof. Since the intervals I appearing in the definition of B_δ are open, it is easily checked that the set B_δ is closed.

By definition of B_δ, for any $\delta' \in (0,\delta)$, the family of intervals I satisfying $\nu(I) > M_h(R\cdot \mathbb{1}_I)$ and $|I| < \delta'$ constitutes a Vitali covering of B_δ to which the Vitali covering theorem can be applied (see [Mat99, Theorem 2.2]). The theorem yields a countable disjoint family $\{I_j\}_j$ of intervals, each of length less than δ', such that

$$\left|B_\delta \setminus \left(\cup_j I_j\right)\right| = 0$$

and $\nu(I_j) > M_h(R\cdot \mathbb{1}_{I_j})$. By definition of M_h, for all j there exists a countable family of open intervals $\{I_{j,k}\}_k$, which we may assume are contained in I_j, such that

$$\sum_k h(|I_{j,k}|) + \int_{I_j \setminus \cup_k I_{j,k}} R\,dm < \nu(I_j). \tag{2.1}$$

Since $\{I_j\}_j$ covers B_δ up to a subset of Lebesgue measure zero, it follows from (2.1) and the disjointness of I_j that

$$\int_{B_\delta \setminus (\cup_{j,k} I_{j,k})} R\,dm < \sum_j \nu(I_j) \le \nu(\mathbb{T}). \tag{2.2}$$

Moreover, we also have $I_{j,k} \subset I_j$ and so $|I_{j,k}| < \delta'$ for every pair of indices j,k. Hence, since $x \mapsto h(x)/x$ is decreasing by assumption, we obtain

$$\sum_{j,k} |I_{j,k}| = \sum_{j,k} h(|I_{j,k}|)\frac{|I_{j,k}|}{h(|I_{j,k}|)} \le \frac{\delta'}{h(\delta')}\sum_{j,k} h(|I_{j,k}|) < \frac{\delta'}{h(\delta')}\nu(\mathbb{T}).$$

By letting $\delta' \to 0$ we obtain that $|\cup_{j,k} I_{j,k}| \to 0$, and it follows from (2.2) that

$$\int_{B_\delta} R\,dm \le \nu(\mathbb{T}). \tag{2.3}$$

Let $\{\ell_j\}_j$ be those intervals in $\mathrm{comp}(B_\delta)$ which satisfy $|\ell_j| < \delta$. Then since the endpoints of ℓ_j are members of B_δ, any interval ℓ_j' of length smaller than δ but strictly containing ℓ_j must satisfy $\nu(\ell_j') > M_h(R\cdot \mathbb{1}_{\ell_j'}) \ge M_h(R\cdot \mathbb{1}_{\ell_j})$. Letting ℓ_j' shrink to ℓ_j and using that ν is atomless we easily obtain $\nu(\ell_j) \ge M_h(R\cdot \mathbb{1}_{\ell_j})$. By definition of M_h, there exists a disjoint family $\{\ell_{j,k}\}_k$ of open intervals contained in ℓ_j such that

$$\sum_k h(|\ell_{j,k}|) + \int_{\ell_j - (\cup_k \ell_{j,k})} R\,dm < \nu(\ell_j) + 2^{-j}. \tag{2.4}$$

Now, let E be the complement in $\mathbb{T}$ of the union of the intervals $\ell_{j,k}$ and the (finitely many) intervals in $\mathrm{comp}(B_\delta)$ which are of length at least δ. Then $B_\delta \subset E$. By (2.4) we see that $\sum_{j,k} h(|\ell_{j,k}|) < \nu(\mathbb{T}) + 1$. Hence $E \in \mathcal{BC}_h$, since $\mathrm{comp}(E)$ includes only finitely many intervals other than $\ell_{j,k}$. Moreover, from (2.4) and (2.3) we also see that

$$\int_E R\,dm = \int_{B_\delta} R\,dm + \sum_j \int_{\ell_j - (\cup_k \ell_{j,k})} R\,dm \le \nu(\mathbb{T}) + \sum_j \nu(\ell_j) + 1 \le 2\nu(\mathbb{T}) + 1.$$

Thus E is the desired set. □

2.2. **The final part of the proof of Theorem B.** Having established the embedding lemma, we may now finish the proof.

Proof of Theorem B. Let ν be a non-negative singular Borel measure which vanishes on all sets $E \in \mathrm{Assoc}\mathcal{BC}_h(w)$. Then ν is certainly atomless. For some sequence $\{\alpha_j\}_j$ of positive numbers which tend to 0, let $B := \cup B_{\alpha_j}$ be the union of sets appearing in the statement of Lemma 2.1. According to the lemma, each of the sets B_{α_j} can be embedded inside a set $E_j \in \mathcal{BC}_h$ for which $\int_{E_j} R\,dm < \infty$, i.e, $E_j \in \mathrm{Assoc}\mathcal{BC}_h(w)$. Since ν is non-negative, we obtain $\nu(B_{\alpha_j}) \leq \nu(E_j) = 0$, by assumption. Thus $\nu(B) = 0$, and so $G := \mathbb{T} \setminus B$ is a set of full ν-measure on $\mathbb{T}$. For every point $z \in G$ we have that $z \notin \cup_j B_{\alpha_j}$, which by definition means that there exist intervals I of length arbitrarily small that contain z and satisfy the inequality $\nu(I) \leq M_h(R \cdot \mathbb{1}_I)$. Since the set G defined above has full ν-measure, the generalized Vitali covering theorem (see [Mat99, Theorem 2.8]) shows that there exists a disjoint family $\{I_j\}_j$ in which each interval I_j has length at most δ, and $\sum_j \nu(I_j) > \nu(\mathbb{T}) - \epsilon$. Since ν is atomless, we may take I_j to be half-open intervals. Set ν_g to be the restriction of ν to the set $\cup_j I_j$. Complete the disjoint family $\{I_j\}_j$ to a partition $\mathcal{P}$ of $\mathbb{T}$ in which each interval of the partition has length less than δ. It follows that $\nu_g(I) \leq M_h(R \cdot \mathbb{1}_I)$ for every $I \in \mathcal{P}$, and that $\nu_r := \nu - \nu_g$ has total mass less than ϵ. The proof is complete. □

2.3. **The converse result: proof of Theorem D.** We show here that our new Roberts-type decomposition characterizes singular measures vanishing on associated sets.

Proof of Theorem D. Assume that $E \in \mathrm{Assoc}\mathcal{BC}_h(w)$, so $E \in \mathcal{BC}_h$ and $\int_E R\,dm < \infty$. Without loss of generality we may assume from the start that ν is singular with respect to Lebesgue measure. Then we must show that $\nu(E) = 0$. There exists a Borel set C of Lebesgue measure zero for which $\nu(E) = \nu(E \cap C)$ and by regularity the last quantity is equal to the supremum of $\nu(A)$ over closed sets $A \subset E \cap C$. Thus it will suffice to show that $\nu(A) = 0$ for all such sets A. Fix one of these sets A for the rest of the proof. We will work with the sets

$$D_\delta = A_\delta \cap E$$

where A_δ is the usual closed δ-neighborhood of A, namely $\{z \in \mathbb{T} : \mathrm{dist}(z, A) \leq \delta\}$. We note the following facts.

- A is a subset of D_δ and $\nu(A) = \lim_{\delta \to 0+} \nu(D_\delta)$ by continuity from above for finite Borel measures.
- We have $\lim_{\delta \to 0} \int_{D_\delta} R\,dm = 0$. This follows by the dominated convergence theorem of measure theory, since $\lim_{\delta \to 0} |D_\delta| = |A| = 0$ and $R \cdot \mathbb{1}_{D_\delta} \leq R \cdot \mathbb{1}_E$.
- We have $D_\delta \in \mathcal{BC}_h$ for every $\delta > 0$. Indeed, A_δ is a finite union of intervals, and so $A_\delta \in \mathcal{BC}_h$. Then $D_\delta = A_\delta \cap E \in \mathcal{BC}_h$ since the family $\mathcal{BC}_h$ is closed under intersections.

Fix $\epsilon > 0$ and let $\nu = \nu_g + \nu_r$ be a decomposition corresponding to a partition $\mathcal{P}$ of $\mathbb{T}$ consisting of intervals of length at most ϵ, $\nu_g(I) \leq M_h(R \cdot \mathbb{1}_I)$ for $I \in \mathcal{P}$, and $\nu_r(\mathbb{T}) \leq \epsilon$. Let $\mathcal{P}_{D_\delta}$ be the subset of $\mathcal{P}$ consisting of those intervals I for which $I \cap D_\delta$ is not empty. The definition of M_h in (1.1) as an infimum shows that for $I \in \mathcal{P}_{D_\delta}$ we have

$$\nu_g(I) \leq M_h(R \cdot \mathbb{1}_I) \leq \sum_{\ell \in \mathrm{comp}(D_\delta) : \ell \cap I \neq \varnothing} h(|\ell \cap I|) + \int_{D_\delta \cap I} R\,dm.$$

Summing over $I \in \mathcal{P}_{D_\delta}$ gives

$$\nu_g(D_\delta) \leq \Big(\sum_{I \in \mathcal{P}_{D_\delta}} \sum_{\ell \in \mathrm{comp}(D_\delta) : \ell \cap I \neq \varnothing} h(|\ell \cap I|) \Big) + \int_{D_\delta} R\,dm. \tag{2.5}$$

Let us estimate the appearing double sum. Since the intervals I in the family $\mathcal{P}_{D_\delta}$ are disjoint, each interval $\ell \in \mathrm{comp}(D_\delta)$ is either contained in some interval in $\mathcal{P}_{D_\delta}$, or it intersects at most two distinct intervals in $\mathcal{P}_{D_\delta}$. In the first case, the term corresponding to ℓ in the double sum equals $h(|\ell \cap I|) = h(|\ell|)$ and we then must have $|\ell| \leq |I| \leq \epsilon$. In the second case, the double sum contains two terms $h(|\ell \cap I_1|)$, $h(|\ell \cap I_2|)$ corresponding to ℓ. Since $|I_1|, |I_2| \leq \epsilon$ and h is increasing, the sum of those terms is bounded by

$$h(|\ell \cap I_1|) + h(|\ell \cap I_2|) \leq 2 \min\{h(|\ell|), h(\epsilon)\}.$$

It follows that the double sum is bounded by

$$\sum_{I \in \mathcal{P}_{D_\delta}} \sum_{\ell \in \mathrm{comp}(D_\delta): \ell \cap I \neq \varnothing} h(|\ell \cap I|) \leq \sum_{\ell \in \mathrm{comp}(D_\delta): |\ell| \leq \epsilon} h(|\ell|) + \sum_{\ell \in \mathrm{comp}(D_\delta)} 2 \min\{h(|\ell|), h(\epsilon)\}.$$

Since $D_\delta \in \mathcal{BC}_h$, we see that for any $\delta > 0$, the sums on the right-hand side tend to 0 as $\epsilon \to 0$. Thus (2.5) gives us

$$\nu(D_\delta) = \nu_g(D_\delta) + \nu_r(D_\delta) \leq o_\epsilon(1) + \int_{D_\delta} R \, dm + \epsilon.$$

Letting $\epsilon \to 0$ we obtain

$$\nu(D_\delta) = \int_{D_\delta} R \, dm.$$

Now letting $\delta \to 0$ we obtain $\nu(A) = 0$ by the first and second points above. By the initial remarks, the proof is now complete. □

3. Proof of Theorem A

3.1. **Hausdorff functionals and their duality.** We need to give a short account of the main optimization theory result from [BM24].

Let $R : \mathbb{T} \to [0, \infty]$ be a non-negative measurable function on $\mathbb{T}$, which we emphasize may attain the value $+\infty$ on a set of positive Lebesgue measure. Consider the family $\mathcal{F}(h, R)$ of all functions $f \in L^\infty(\mathbb{T})$, which satisfy the constraints

$$0 \leq f(z) \leq R(z), \quad z \in \mathbb{T}$$

and

$$\int_I f \, dm \leq h(|I|), \quad I \text{ arc on } \mathbb{T}.$$

We are interested in the maximization of the quantity $\|f\|_1 = \int_{\mathbb{T}} f \, dm$ over $f \in \mathcal{F}(h, R)$. The following is the main result of [BM24].

Theorem 3.1. *If $M_h(R) > 0$, then*

$$6 \cdot \sup_{f \in \mathcal{F}(h,R)} \int_{\mathbb{T}} f \, dm > M_h(R).$$

Here the quantity on the right-hand side of the inequality is just the value $M_h(R)$ of the Hausdorff functional evaluated at R which we introduced earlier. In principle, that the value $M_h(R)$ governs the amount of mass f one can place on $\mathbb{T}$ under the constraint $f \in \mathcal{F}(h, R)$.

3.2. **Deducing Theorem A from Theorem B.** Proof of Theorem A can now be completed essentially as in [Mal26], with the new Roberts-type decomposition replacing the more complicated linear program appearing there.

Proof of Theorem A. Let ν be a singular measure as in the statement of the theorem. Note that the class of sets $\mathrm{Assoc}\mathcal{BC}_h(w)$ is invariant under replacing h by a positive scalar multiple of itself and under replacing w by a positive power of itself. Set $\epsilon = \delta$ in Theorem B and apply the theorem to the gauge function $h/6$ and the weight $w^{1/6}$. Set $R = \max\{\log(1/w), 0\}$, as before. We obtain a partition $\mathcal{P}$ into intervals of lengths at most ϵ, a decomposition $\nu = \nu_g + \nu_r$ and estimates $\nu_g(I) \le M_{h/6}(R/6 \cdot \mathbb{1}_I)$, $I \in \mathcal{P}$. All of these objects depend on the parameter $\epsilon > 0$, but we suppress this in our notation for now. By the duality in Theorem 3.1 applied to the functions $R/6 \cdot \mathbb{1}_I$, for each interval $I \in \mathcal{P}$ we obtain a bounded non-negative measurable function f_I satisfying $\int_I f_I \, dm = \nu_g(I)$, $\int_\Delta f_I \, dm \le h(|\Delta|)$ for every interval $\Delta \subset \mathbb{T}$, and $f_I \le R \cdot \mathbb{1}_I$ on $\mathbb{T}$[4]. Consider the non-negative function $f = \sum_{I \in \mathcal{P}} f_I$. This function satisfies the following three properties:

(i) $f \le R$ on $\mathbb{T}$,
(ii) $\int_I f \, dm = \nu_g(I)$ for all $I \in \mathcal{P}$,
(iii) $-\nu_g(\Delta) + \int_\Delta f \, dm \le 2h(|\Delta|)$ for all intervals $\Delta \subset \mathbb{T}$.

Of these three properties, only the last one is not a direct consequence of the properties of the functions f_I. To see that it holds, note that there exist at most two intervals I in the partition $\mathcal{P}$ which contain the endpoints of Δ. If both endpoints are contained in the same $I \in \mathcal{P}$, then

$$-\nu_g(\Delta) + \int_\Delta f \, dm \le \int_\Delta f_I \, dm \le h(|\Delta|) \le 2h(|\Delta|).$$

Otherwise, let the two endpoints be contained in I_L and I_R. Then, by property (ii), we have

$$-\nu_g(\Delta) + \int_\Delta f \, dm = -\nu_g(I_L \cap \Delta) - \nu_g(I_R \cap \Delta) + \int_{I_L \cap \Delta} f \, dm + \int_{I_R \cap \Delta} f \, dm.$$

Since ν_g is a non-negative measure, $f = f_{I_L}$ on I_L and $f = f_{I_R}$ on I_R, we obtain

$$-\nu_g(\Delta) + \int_\Delta f \, dm \le \int_{I_L \cap \Delta} f_{I_L} \, dm + \int_{I_R \cap \Delta} f_{I_R} \, dm \le h(|I_L \cap \Delta|) + h(|I_R \cap \Delta|) \le 2h(|\Delta|).$$

Consider the measure $dM_\epsilon := -d\nu + f \cdot dm = -d\nu_r + (-d\nu_g + f \cdot dm)$, where $\epsilon > 0$ is the parameter chosen earlier on which the construction of ν_g, ν_r and f depends. It is not hard to see that $M_\epsilon \to 0$ in the weak sense of measures, as $\epsilon \to 0$. Indeed, by property (ii) above the total variation of M_ϵ is $|M_\epsilon|(\mathbb{T}) \le 2\nu(\mathbb{T})$, and we have also $\nu_r(\mathbb{T}) \le \epsilon$. Since $\sup\{|I| : I \in \mathcal{P}\} \le \epsilon$, we readily obtain that

$$\int_{\mathbb{T}} \phi(d\nu_g - f \cdot dm) = \sum_{I \in \mathcal{P}} \int_I \phi(d\nu_g - f \cdot dm) \to 0$$

for any ϕ continuous on $\mathbb{T}$. Hence indeed $M_\epsilon \to 0$ weakly.

Since ν_r is positive, we obtain by property (iii) above that

$$M_\epsilon(\Delta) \le 2h(|\Delta|)$$

for all intervals $\Delta \subset \mathbb{T}$. This implies that the Poisson integral of M_ϵ, namely

$$U_\epsilon(z) := \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} \, dM_\epsilon(\zeta), \quad z \in \mathbb{D},$$

[4]If $M_{h/6}(R/6 \cdot \mathbb{1}_I) = 0$ then we simply set $f_I = 0$.

satisfies an upper growth estimate of the form

$$U_\epsilon(z) \le C\frac{h(1-|z|)}{1-|z|}, \quad z \in \mathbb{D}$$

for some (computable) numerical constant $C > 0$. This growth bound is well-known, see for instance [HJ12, page 297]. By invariance of our problem for the change $Ch(x) \to h(x)$, we may assume for convenience that $C = 1$. By the mentioned weak convergence, we have $U_\epsilon(z) \to 0$ as $\epsilon \to 0$. Let $\widetilde{U_\epsilon}$ be the harmonic conjugate of U_ϵ in $\mathbb{D}$, normalized by $\widetilde{U_\epsilon}(0) = 0$. Then the structure of the measure M_ϵ says that

$$\exp\big(U_\epsilon(z) + i\widetilde{U_\epsilon}(z)\big) = S_\nu(z)g_\epsilon(z)$$

where $g_\epsilon \in H^\infty(\mathbb{D})$ satisfies $|g_\epsilon| = \exp(f) \le \exp(R) \le \max(1/w, 1)$ a.e. on $\mathbb{T}$. Thus $\|S_\nu g_\epsilon w\|_{\infty,\mathbb{T}} \le 1$. By weak convergence, we obtain $S_\nu(z)g_\epsilon(z) \to 1$ as $\epsilon \to 0$. Moreover,

$$|S_\nu(z)g_\epsilon(z)| = \exp(U_\epsilon(z)) \le \exp\Big(\frac{h(1-|z|)}{1-|z|}\Big)$$

by our earlier pointwise growth estimate on U_ϵ. Choosing a sequence $(\epsilon_n)_n$ tending to 0 and setting $g_n = g_{\epsilon_n}$ completes the proof. □

References

[BM24] L. Bergqvist and B. Malman. Distributing mass under a pointwise bound and an application to weighted polynomial approximation. *arXiv preprint arXiv:2408.05222*, 2024.

[HJ12] V. Havin and B. Jöricke. *The uncertainty principle in harmonic analysis*, volume 28. Springer Science & Business Media, 2012.

[IN24] O. Ivrii and A. Nicolau. Beurling–carleson sets, inner functions and a semilinear equation. *Analysis & PDE*, 17(7):2585–2618, 2024.

[Ivr19] O. Ivrii. Prescribing inner parts of derivatives of inner functions. *Journal d'Analyse Mathématique*, 139(2):495–519, 2019.

[KM90] T. L. Kriete and B. D. MacCluer. Mean-square approximation by polynomials on the unit disk. *Transactions of the American Mathematical Society*, 322(1):1–34, 1990.

[Kor75] B. Korenblum. An extension of the Nevanlinna theory. *Acta Mathematica*, 135:187–219, 1975.

[Kor77] B. Korenblum. A Beurling-type theorem. *Acta Mathematica*, 138(1):265–293, 1977.

[Kor81] B. Korenblum. Cyclic elements in some spaces of analytic functions. *Bulletin of the American Mathematical Society*, 5(3):317–318, 1981.

[Lim24] A. Limani. Shift invariant subspaces in growth spaces and sets of finite entropy. *Journal of the London Mathematical Society*, 109(5):e12915, 2024.

[LM23] A. Limani and B. Malman. On the problem of smooth approximations in $\mathcal{H}(b)$ and connections to subnormal operators. *Journal of Functional Analysis*, 284(5):109803, 2023.

[Mal26] B. Malman. Weighted Korenblum-Roberts theory. *Journal of Functional Analysis*, 290(7), 2026.

[Mat99] P. Mattila. *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability*. Number 44. Cambridge university press, 1999.

[Rob85] J. Roberts. Cyclic inner functions in the Bergman spaces and weak outer functions in H^p, $0 < p < 1$. *Illinois Journal of Mathematics*, 29(1):25–38, 1985.

Institutionen för ekonomi och matematik, Mälardalens Universitet, 721 23 Västerås, Sweden
Email address: bartosz.malman@mdu.se