POLAR CARLESON ZERO SET PREVENTING DIRICHLET CYCLICITY

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Abstract. The work [CFG26] of Chen, Fang and Guo shows, finally, that the Brown-Shields conjecture is false. In this note[1], we construct a polar Carleson set E on the unit circle $\mathbb{T}$ which has the property that any Hölder continuous function in the Dirichlet space vanishing on E is non-cyclic.

1. **Reduction to an estimate of a point evaluation and a discussion.** Let $\mathcal{D}$ be the Dirichlet space on the unit disk $\mathbb{D}$, equipped with the norm $\|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z)$, where dA is normalized area measure. On $\mathbb{T}$, let m be normalized Lebesgue measure and write $|B| = m(B)$ for every measurable set $B \subseteq \mathbb{T}$.

Proposition 1. *There exist a compact Carleson set $E \subset \mathbb{T}$ of zero logarithmic capacity and a numerical constant $C > 0$ with the following property. If $h \in C(\mathbb{T}) \cap \mathcal{D}$ vanishes on E and satisfies the distance estimate*

$$|h(t)| \leq C_h \, dist(t, E)^{1/2}, \quad t \in \mathbb{T},$$

then

$$|h(0)|^2 \leq C \int_{\mathbb{D}} |h'(z)|^2 \, dA(z). \tag{2}$$

In particular, the estimate in (2) *does not involve the constant C_h.*[2]

The definition of Carleson sets is given below, right after (10). We spend the note on proving Proposition 1.

For $\alpha \in (0, 1)$, let $C^\alpha(\mathbb{T})$ be the space of Hölder regular functions on $\mathbb{T}$ with exponent α, namely functions satisfying an estimate of the form $|f(t) - f(s)| \leq C_{f,\alpha}|t - s|^\alpha$ for some constant $C_{f,\alpha} > 0$. Then the distance estimate in Proposition 1 is satisfied for $h \in C^{1/2}(\mathbb{T})$ that vanishes on E. If $h \in C^\alpha(\mathbb{T})$ vanishes on E and the integer k satisfies $k\alpha \geq 1/2$, then the distance estimate is satisfied for h^k and ph^k, where p is any polynomial.

Corollary 2. *Every outer function $f \in C^\alpha(\mathbb{T}) \cap \mathcal{D}$ that vanishes on the set E in Proposition* 1 *is non-cyclic in $\mathcal{D}$.*

Proof. Let k be an integer satisfying $k\alpha \geq 1/2$, and set $g = f^k$. Since f is bounded, we have $g \in \mathcal{D}$. By the paragraph preceding the corollary, we have

$$|p(0)g(0)|^2 \leq C \int_{\mathbb{D}} |(p(z)g(z))'|^2 dA(z)$$

for any polynomial p. Since

$$\int_{\mathbb{D}} |(p(z)g(z))'|^2 dA(z) = \int_{\mathbb{D}} |(p(z)g(z) - 1)'|^2 dA(z) \leq \|pg - 1\|_{\mathcal{D}}^2$$

we deduce that

$$|p(0)g(0)|^2 \leq C\|pg - 1\|_{\mathcal{D}}^2$$

and we see that we cannot have $pg \to 1$ in $\mathcal{D}$, for then the left-hand side above tends to 1 while the right hand side tends to 0. Hence g is non-cyclic. By [EKMR14, Theorem 9.1.7], $g = f^k$ is cyclic if and only if f is cyclic. □

Given a Carleson set E with $|E| = 0$, there exists an outer function $f \in C^{1/2}(\mathbb{T}) \cap \mathcal{D}$ whose boundary zero set $\{z \in \mathbb{T} : f(z) = 0\}$ is exactly E. See [EKMR14, Theorem 4.4.3], or [TW70] for examples in $C^\infty(\mathbb{T}) \cap \mathcal{D}$. For the particular E constructed in Proposition 1, any such function is non-cyclic in $\mathcal{D}$. This should be contrasted with the construction of Brown and Cohn in [BC85] that exhibits a cyclic function in $C(\mathbb{T}) \cap \mathcal{D}$ which vanishes on any prescribed compact polar set $E \subset \mathbb{T}$. Thus the same set E is the exact boundary zero set of two continuous

[1] All substantive results in this note have been given by the Oracle, in this case the ChatGPT 6.0 Astra model. But she did not write the note.

[2] We mention, at once, that the magical disappearance of dependence on C_h occurs in the limit as $p \to \infty$ in (13) in which an iterative inequality associated to E is employed.

outer Dirichlet functions, one cyclic and one non-cyclic. It follows that Corollary 2 fails if $C^\alpha(\mathbb{T})$ is replaced by the larger class $C(\mathbb{T})$.

2. **The heart of the matter.**

2.1. *Carleson integrals.* For an arc $I \subseteq \mathbb{T}$ and a nonzero function $f \in \mathcal{D}$, consider the Carleson integral over I, namely

$$\mathrm{Carl}_I(f) = \int_I \int_{\mathbb{T}} \frac{(|f(t)|^2 - |f(s)|^2)(\log|f(t)|^2 - \log|f(s)|^2)}{|t-s|^2}\, dm(s)\, dm(t). \tag{3}$$

According to [EKMR14, Theorem 7.4.1], for outer functions f we have the identity $\mathrm{Carl}_{\mathbb{T}}(f) = 2\int_{\mathbb{D}} |f'|^2\, dA$. We remark that we will use the Carleson integrals above even for functions $f \in \mathcal{D}$ which are not outer. Indeed, the estimate in Proposition 3 below applies anyway. This will play a role, albeit minor.

2.2. *The main estimate.* For a family $\mathcal{I}$ of intervals on $\mathbb{T}$ with disjoint interiors, we introduce a notation for the sum of averages of $|f|^2$ over the family:

$$\mathcal{T}_{\mathcal{I}}(f) = \sum_{I\in\mathcal{I}} \frac{1}{|I|}\int_I |f|^2\, dm. \tag{4}$$

The next proposition is a variant of [CFG26, Theorem 11.1]. It may be beneficial to keep referring to Figure 1 during the reading of the following parts in the red box.

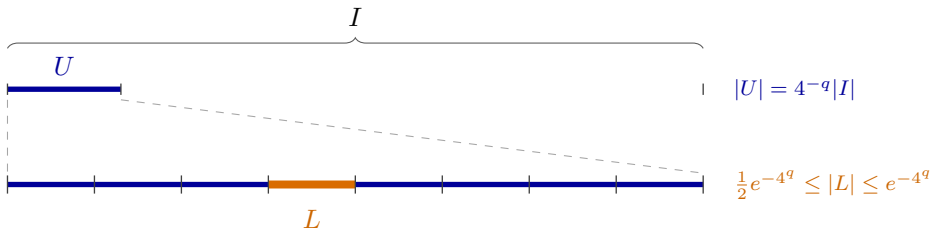


FIGURE 1. The lower row shows the subdivision of U. In the construction of our Carleson set E, the same procedure will be recursively applied to each L.

Proposition 3. *Let $I \subseteq \mathbb{T}$ be an interval and let q be a large positive integer. Let U be the leftmost subinterval of I of length $4^{-q}|I|$. Divide U into equal-length intervals L satisfying*

$$\frac{e^{-4^q}}{2} \le |L| \le e^{-4^q}. \tag{5}$$

Let the finite family $\mathcal{I}'$ consist of all the intervals L of this division. Then, for every $f \in C(\mathbb{T}) \cap \mathcal{D}$, we have

$$\frac{1}{|I|}\int_I |f|^2\, dm \le 16\pi^2 \cdot \mathrm{Carl}_I(f) + \frac{e^{-4^q/2}4^q}{|I|}\mathcal{T}_{\mathcal{I}'}(f). \tag{6}$$

Proof. Note first that for $a, b > 0$ we have

$$a \le \frac{4}{4^q}(a-b)(\log a - \log b) + e^{4^q/2}b. \tag{7}$$

Indeed, if $a \le e^{4^q/2}b$, the inequality is clear, for the first term on the right-hand side is non-negative. Otherwise $a > e^{4^q/2}b$, and by taking logarithms and rearranging it follows that $\frac{2(\log a - \log b)}{4^q} > 1$. Since $e^{4^q/2} > 2$, the inequality $a > e^{4^q/2}b$ also implies $2(a-b) \ge a$. Then (7) follows by multiplying the two derived inequalities.

In (7) take $a = |f(t)|^2$ and $b = |f(s)|^2$, and integrate over $t \in I$ and $s \in U$. We obtain

$$|U|\int_I |f|^2 dm \le \frac{4}{4^q}\int_I\int_U (|f(t)|^2 - |f(s)|^2)(\log|f(t)|^2 - \log|f(s)|^2)\, dm(s)\, dm(t) + e^{4^q/2}|I|\int_U |f|^2 dm. \tag{8}$$

In the first integral term on the right-hand side, we multiply the integrand by $\frac{4\pi^2|I|^2}{|t-s|^2}$ which is greater than 1 since $|t-s| < 2\pi|I|$ as $t, s \in I$, and then we extend the s-integral from U to $\mathbb{T}$ by non-negativity of the integrand. For the second integral term, we use that U is the union of the intervals $L \in \mathcal{I}'$ and (5) to obtain

$$e^{4^q/2}|I|\int_U |f|^2 dm = e^{4^q/2}|I|\sum_{L\in\mathcal{I}'}\int_L |f|^2 dm \le e^{-4^q/2}|I|\mathcal{T}_{\mathcal{I}'}(f).$$

Putting these estimates into (8) and dividing by $|I|\cdot|U|$ gives

$$\begin{aligned}\frac{1}{|I|}\int_I |f|^2\, dm &\le \frac{16\pi^2|I|}{4^q\cdot|U|}\operatorname{Carl}_I(f) + \frac{e^{-4^q/2}}{|U|}\mathcal{T}_{\mathcal{I}'}(f)\\ &= 16\pi^2\operatorname{Carl}_I(f) + \frac{e^{-4^q/2}4^q}{|I|}\mathcal{T}_{\mathcal{I}'}(f).\end{aligned}$$

□

Corollary 4. *Let $\mathcal{I}=\{I_1,\dots,I_s\}$ be a finite family of intervals in $\mathbb{T}$ of equal length with disjoint interiors. For each $I_j\in\mathcal{I}$, let $\mathcal{I}'_j(q)$ be the interval family constructed in Proposition 3, with common interval length across all those families, and let $\mathcal{I}'(q)$ be the union of those families. For all q sufficiently large, it holds that for any $f\in C(\mathbb{T})\cap\mathcal{D}$, we have*

$$\mathcal{T}_{\mathcal{I}}(f) \le 16\pi^2\operatorname{Carl}_{\mathbb{T}}(f) + \frac{1}{4}\mathcal{T}_{\mathcal{I}'(q)}(f). \tag{9}$$

Given $\epsilon>0$, for all sufficiently large q we have that the capacity of $\bigcup_{L\in\mathcal{I}'(q)} L$ is less than ϵ.

Proof. Choose q at least so large that $\frac{e^{-4^q/2}4^q}{|I_j|}<1/4$. The intervals I_j have disjoint interiors, so $\sum_{j=1}^s \operatorname{Carl}_{I_j}(f)\le \operatorname{Carl}_{\mathbb{T}}(f)$ by additivity of (3). Summing (6) over j gives (9). The family $\mathcal{I}'_j(q)$ consists of intervals contained within a common interval U_j of length $4^{-q}|I_j|$. Thus the capacity of $\cup_{L\in\mathcal{I}'_j(q)}L$ tends to 0 as $q\to\infty$. The capacity claim then follows, since there are only finitely many families $\mathcal{I}'_j(q)$. □

3. The construction of the Carleson set E.

3.1. *Carleson sets.* For a finite or countable set of arcs $\mathcal{G}$, the Carleson entropy is

$$\kappa(\mathcal{G}) = \sum_{G\in\mathcal{G}} |G|\log(e/|G|). \tag{10}$$

A compact set $E\subset\mathbb{T}$ with $|E|=0$ is a Carleson set if $\kappa(\mathcal{G}(E))<\infty$, where $\mathcal{G}(E)$ is the family of maximal connected intervals constituting the complement $\mathbb{T}\setminus E$.

We have the **merging law**: if each interval in $\mathcal{G}'$ is a disjoint union of intervals in $\mathcal{G}$ (say, up to a set of measure zero, and each interval in $\mathcal{G}$ being used in merging only once), then $\kappa(\mathcal{G}')\le\kappa(\mathcal{G})$. This follows immediately from the inequality $\phi(a+b)\le\phi(a)+\phi(b)$ for $\phi(t)=t\log(e/t)$.

3.2. *Proof of Proposition* 1. We construct E as an intersection of unions of intervals in the families appearing in Corollary 4, and iterate the estimate in (9) to prove (2). Finally, we verify that E is Carleson. The construction is a variant of the one given in [CFG26, Section 11.3].

Set $\mathcal{I}_1=\{\mathbb{T}\}$. At each step $n\ge 2$, apply[3] Corollary 4 to $\mathcal{I}_{n-1}=\mathcal{I}$ to obtain $\mathcal{I}_n=\mathcal{I}'(q_n)$, choosing q_n so that the capacity of $\bigcup_{L\in\mathcal{I}_n} L$ tends to 0, and so that we have

$$\mathcal{T}_{\mathcal{I}_{n-1}}(f) \le 16\pi^2\operatorname{Carl}_{\mathbb{T}}(f) + \frac{1}{4}\mathcal{T}_{\mathcal{I}_n}(f) \tag{11}$$

If we take (as we may) the intervals L to be closed, then

$$E = \bigcap_{n\ge1}\Big(\bigcup_{L\in\mathcal{I}_n} L\Big)$$

is clearly a closed polar set. The left endpoint of each $L\in\mathcal{I}_n$ survives to E. This follows immediately from the construction of the families $\mathcal{I}_n$.

Let us prove (2) for h as stated in the proposition. If a function $h\in C(\mathbb{T})$ satisfies $|h(t)|\le C_h\cdot\operatorname{dist}(t,E)^{1/2}$ for some constant $C_h>0$, then since each $L\in\mathcal{I}_n$ intersects E, we have $\operatorname{dist}(t,E)\le 2\pi|L|$ for $t\in L$, and consequently $|h(t)|\le C_h 2\pi\sqrt{|L|}$ for $t\in L$. According to (4), we obtain

$$\mathcal{T}_{\mathcal{I}_n}(h) \le \sum_{L\in\mathcal{I}_n}\frac{1}{|L|}\int_L C_h^2 4\pi^2|L|\, dm \le C_h^2 4\pi^2\sum_{L\in\mathcal{I}_n}|L| \le 4\pi^2 C_h^2. \tag{12}$$

[3] At step $n=2$ we have $I=\mathbb{T}$ and so we have no endpoints. Take U in Proposition 3 equal to $\{e^{i2\pi x}\in\mathbb{T} : x\in[0,4^{-q_2}]\}$.

The last inequality follows from disjointness of the interiors of the intervals L and $|\mathbb{T}| = 1$. Iterating (11) and using (12) we obtain that for any $p \geq 2$ we have

$$\begin{aligned} \int_{\mathbb{T}} |h|^2 \, dm &\leq 16\pi^2 \left(\sum_{j=0}^{p-2} 4^{-j} \right) \operatorname{Carl}_{\mathbb{T}}(h) + \frac{\mathcal{T}_{\mathcal{I}_p}(h)}{4^{p-1}} \\ &\leq C \operatorname{Carl}_{\mathbb{T}}(h) + \frac{4\pi^2 C_h^2}{4^{p-1}} \end{aligned} \tag{13}$$

with C some (finite) numerical constant. Letting $p \to \infty$ in (13) we obtain $\int_{\mathbb{T}} |h|^2 \, dm \leq C \operatorname{Carl}_{\mathbb{T}}(h)$. If $h = IQ$ is the inner-outer factorization of h into inner factor I and outer factor Q, then $|h| = |Q|$ almost everywhere on $\mathbb{T}$, and so $\int_{\mathbb{T}} |h|^2 \, dm = \int_{\mathbb{T}} |Q|^2 \, dm$ and $\operatorname{Carl}_{\mathbb{T}}(h) = \operatorname{Carl}_{\mathbb{T}}(Q)$. Hence, using also that $\operatorname{Carl}_{\mathbb{T}}(Q) = 2 \int_{\mathbb{D}} |Q'|^2 \, dA$ for the outer function Q, we obtain

$$\begin{aligned} |h(0)|^2 \leq \int_{\mathbb{T}} |h|^2 \, dm &\leq C \operatorname{Carl}_{\mathbb{T}}(Q) \\ &= 2C \int_{\mathbb{D}} |Q'(z)|^2 \, dA(z) \\ &\leq 2C \int_{\mathbb{D}} |h'(z)|^2 \, dA(z). \end{aligned}$$

The second line uses the remark following (3), and the last line uses that division by an inner factor can only decrease the Dirichlet integral [EKMR14, Corollary 7.6.3]. This proves (2).

It remains to verify that E indeed is a Carleson set. For $I \in \mathcal{I}_n$ we have according to Figure 1 that $I \setminus \left(\bigcup_{I' \in \mathcal{I}_{n+1}, I' \subset I} I' \right) = I \setminus U$. Each interval in $\mathcal{G}(E)$ is, up to endpoints, a countable disjoint union of the intervals $I \setminus U$, $I \in \mathcal{I}_n$, $n \geq 1$. By the merging law, it suffices to bound the sum of the Carleson entropies of the intervals $I \setminus U$. For every I, the Carleson entropy of $I \setminus U$ is $|I \setminus U| \log(e/|I \setminus U|) \leq |I| \log(e/|I|)$.

Let N_n be the number of intervals in $\mathcal{I}_n$, and let ℓ_n be their common length. By the above paragraph, each $I \in \mathcal{I}_n$ contributes not more than $\ell_n \log(e/\ell_n)$ to the entropy of $\mathcal{G}(E)$. In order to show that E is a Carleson set, it thus remains to show that

$$\sum_{n \geq 1} N_n \ell_n \log(e/\ell_n) < \infty. \tag{14}$$

For $n \geq 2$, the ratio of $N_n \ell_n$ to $N_{n-1} \ell_{n-1}$ is the fraction of each parent interval $I \in \mathcal{I}_{n-1}$ covered by the blue interval U in Figure 1. Thus

$$\frac{N_n \ell_n}{N_{n-1} \ell_{n-1}} = 4^{-q_n}. \tag{15}$$

By (5) we have $\ell_n \geq e^{-4^{q_n}}/2$, so

$$\log(e/\ell_n) \leq 2 + 4^{q_n} \leq 2 \cdot 4^{q_n}.$$

Consequently,

$$N_n \ell_n \log(e/\ell_n) \leq 2 N_{n-1} \ell_{n-1}.$$

But by (15), $N_n \ell_n$ is certainly geometrically decreasing in n. Thus the last displayed inequality shows that (14) holds. The proof is complete.

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